

PERTURBATIONS OF CROSSED PRODUCT C*-ALGEBRAS BY AMENABLE GROUPS

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ABSTRACT. We study uniform perturbations of crossed product C*-algebras by amenable groups. Given a unital inclusion of C*-algebras $C \subseteq D$ and sufficiently close separable intermediate C*-subalgebras A, B for this inclusion with a conditional expectation from D onto B , if $A = C \rtimes G$ with G discrete amenable, then A and B are isomorphic. Furthermore, if $C \subseteq D$ is irreducible, then $A = B$.

1. INTRODUCTION

Kadison and Kastler started the study of perturbation theory of operator algebras with [15] in 1972. They equipped the set of operator algebras on a fixed Hilbert space with a metric induced by Hausdorff distance between the unit balls. Examples of close operator algebras are obtained by conjugating by a unitary near to the identity. They conjectured sufficiently close operator algebras must be unitarily equivalent. For injective von Neumann algebras, this conjecture was settled in [6, 22, 13, 8] with earlier special cases [5, 18]. Cameron et al. [2] and Chan [3] gave examples of non-injective von Neumann algebras satisfying the Kadison-Kastler conjecture. In Christensen [7], this conjecture was solved positively for von Neumann subalgebras of a common finite von Neumann algebra.

For C*-algebras, the separable nuclear case was solved positively in Christensen et al. [9], building on the earlier special cases in [8, 19, 20, 16]. In full generality, there are examples of arbitrarily close non-separable nuclear C*-algebras which are not *-isomorphic in Choi and Christensen [4]. Johnson gave examples of arbitrarily close pairs of separable nuclear C*-algebras which conjugate by unitaries where the implementing unitaries could not be chosen to be the identity in [14].

The author and Watatani [12] showed that for an inclusion of simple C*-algebras $C \subseteq D$ with finite index in the sense of Watatani [23], sufficiently close intermediate C*-subalgebras are unitarily equivalent. The implementing unitary can be chosen close to the identity and in the relative commutant algebra $C' \cap D$. Our estimates depend on the inclusion $C \subseteq D$, since we use the finite basis for $C \subseteq D$. Dickson obtained uniform estimates independent of all inclusions in [10]. To get this, Dickson showed that row metric is equivalent to the Kadison-Kastler metric.

The author [11] showed that von Neumann subalgebras of a common von Neumann algebra with finite probabilistic index in the sense of Pimsner-Popa [21] satisfy the Kadison-Kastler conjecture. The implementing unitary can be chosen as being close to the identity. Compared with the author and Watatani case [12], we do not assume that von Neumann subalgebras have a common subalgebra with finite index.

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In this paper, we study perturbations of crossed product C^* -algebras by discrete amenable groups. We introduce crossed product-like inclusions of C^* -algebras in Definition 3.1. For a unital inclusion of C^* -algebras $A \subseteq B$, we call $A \subseteq B$ is crossed product-like if there exists a discrete group U in the normalizer $\mathcal{N}_B(A)$ of A in B such that A and U generate B . An example of crossed product-like inclusions is $A \subseteq A \rtimes G$, where G is a discrete group.

Now suppose that we have a unital inclusion $C \subseteq D$ of C^* -algebras and two close separable intermediate C^* -subalgebras A, B for this inclusion. If there is a conditional expectation $E: D \rightarrow B$, then we get a map from A into B which is uniformly close to the identity map of A by restricting E to A . Since C is a subalgebra of $A \cap B$, $E|_A: A \rightarrow B$ is a C -fixed map, that is, $E|_A(c) = c$ for any $c \in C$. Furthermore, if $C \subseteq A$ is crossed product-like by a discrete amenable group U in $\mathcal{N}_A(C)$, then we can consider the point-norm averaging technique from [9] by using the amenability of U . To apply this technique to $E|_A$ we need that $E|_A$ is a C -fixed map. Then in Lemma 3.7, we can obtain a C -fixed (X, ε) -approximate $*$ -homomorphism from A into B for a finite subset X in A_1 and $\varepsilon > 0$. To show this, we modify [9, Lemma 3.2] to C -fixed versions. In Lemma 3.8, we obtain unitaries which conjugate these maps by modifying [9, Lemma 3.4] to C -fixed version. The unitaries can be chosen in the relative commutant $C' \cap D$ of C in D . Therefore, if $C \subseteq D$ is irreducible, then the unitaries are scalars. Then by these lemmas, we show our first main result: Theorem A, which is appeared in Theorem 3.10.

Theorem A. *Let $C \subseteq D$ be a unital irreducible inclusion of C^* -algebras acting on a separable Hilbert space H . Let A and B be separable intermediate C^* -subalgebras for $C \subseteq D$ with a conditional expectation from D onto B . Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < 140^{-1}$. Then $A = B$.*

In Theorem B, we show our second main result. By an intertwining argument which is modified [9, Lemma 4.1] to C -fixed version, we show that A is $*$ -isomorphic to B . The implementing surjective $*$ -isomorphism can be chosen as C -fixed. Theorem B is provided in Section 4 as Theorem 4.3.

Theorem B. *Let $C \subseteq D$ be a unital inclusion of C^* -algebras and let A and B be separable intermediate C^* -subalgebras for $C \subseteq D$ with a conditional expectation from D onto B . Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < 10^{-3}$. Then there exists a C -fixed surjective $*$ -isomorphism $\alpha: A \rightarrow B$.*

In section 5, we consider crossed product-like inclusions of von Neumann algebras. Given an inclusion $N \subseteq M$ of von Neumann algebras, we call $N \subseteq M$ is crossed product-like if there is a discrete group U in $\mathcal{N}_M(N)$ such that M is generated by N and U . For a crossed product-like inclusion $A \subseteq B$ of C^* -algebras acting non-degenerately on H , an inclusion $\overline{A}^w \subseteq \overline{B}^w$ of von Neumann algebras is crossed product-like. In Theorem C, we consider the perturbations of crossed product von Neumann algebras by discrete amenable groups. This result is based on Christensen's work [6] and is appeared in Theorem 5.7.

Theorem C. *Let $N \subseteq M$ be an inclusion of von Neumann algebras in $\mathbb{B}(H)$ and let A, B be intermediate von Neumann subalgebras for $N \subseteq M$ with a normal conditional expectation from M onto B . Suppose that $N \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-2}$. Then there exists a unitary $u \in N' \cap (A \cup B)''$ such that $uAu^* = B$ and $\|u - I\| \leq 2(8 + \sqrt{2})\gamma$.*

By the theorem above, we can consider the perturbations of the second dual C*-algebras of crossed product algebras by amenable groups in Corollary 5.8. Given a unital inclusion $C \subseteq D$ of C*-algebras and sufficiently close intermediate C*-subalgebras A, B for this inclusion, if $C \subseteq A$ is a crossed product-like inclusion by a discrete amenable group and there is a conditional expectation $E: D \rightarrow B$, then A^{**} and B^{**} are unitarily equivalent. To show this, we use a normal conditional expectation $E^{**}: D^{**} \rightarrow B^{**}$ and identify A^{**}, B^{**}, C^{**} and D^{**} with $\pi(A)''$, $\pi(B)''$, $\pi(C)''$ and $\pi(D)''$, respectively, where π is the universal representation of D .

In Proposition 6.3, we obtain a unitary such that the unitary implement a *-isomorphism under the assumption $C' \cap C^*(A, B) \subseteq \overline{C'} \cap \overline{A}^w$. To show Proposition 6.3 we prepare Lemma 6.1 and 6.2 by using Lemma 3.8 and 3.12 and Theorem 4.3. Combining Proposition 6.3 with Corollary 5.8 gives Theorem D, which is appeared in Theorem 6.4. To show this, we modify the arguments of Section 5 in Christensen et al. [9].

Theorem D. *Let $C \subseteq D$ be a unital inclusion of C*-algebras acting on a separable Hilbert space H . Let A and B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $C' \cap A$ is weakly dense in $C' \cap \overline{A}^w$. If $d(A, B) < 10^{-7}$, then there exists a unitary $u \in C' \cap (A \cup B)''$ such that $uAu^* = B$.*

2. PRELIMINARIES

Given a C*-algebra A , we denote by A_1 and A^u the unit ball of A and the unitaries in A , respectively. We recall Kadison and Kastler's metric on the set of all C*-subalgebras of a C*-algebra from [15].

Definition 2.1. Let A and B be C*-subalgebras of a C*-algebra C . Then we define a metric between A and B by

$$d(A, B) := \max \left\{ \sup_{a \in A_1} \inf_{b \in B_1} \|a - b\|, \sup_{b \in B_1} \inf_{a \in A_1} \|a - b\| \right\}.$$

In the definition above, $d(A, B) < \gamma$ if and only if for any x in either A_1 or B_1 , there exists y in other one such that $\|x - y\| < \gamma$.

Example 2.2. Let A be a C*-algebra in $\mathbb{B}(H)$ and u be a unitary in $\mathbb{B}(H)$. Then $d(A, uAu^*) \leq 2\|u - I_H\|$.

Near inclusions of C*-algebras are defined by Christensen in [8].

Definition 2.3. Let A and B be C*-subalgebras of a C*-algebra C and let $\gamma > 0$. We write $A \subseteq_\gamma B$ if for any $x \in A_1$ there exists $y \in B$ such that $\|x - y\| \leq \gamma$. If there is $\gamma' < \gamma$ with $A \subseteq_{\gamma'} B$, then we write $A \subset_\gamma B$.

The next two proposition is folklore. The second can be found as [9, Proposition 2.10].

Proposition 2.4. *Let A and B be C*-algebras with $A \subseteq B$. If $B \subset_1 A$, then $A = B$.*

Proposition 2.5. *Let A and B be C*-subalgebras of a C*-algebra C . If $B \subset_{1/2} A$ and A is separable, then B is separable.*

The following lemma appears in [15, Lemma 5].

Lemma 2.6. *Let A and B be C^* -subalgebras acting on a Hilbert space H . Then $d(\overline{A}^w, \overline{B}^w) \leq d(A, B)$.*

The lemma below shows some standard estimates.

Lemma 2.7. *Let A be a unital C^* -algebra.*

- (1) *Given $x \in A$ with $\|I - x\| < 1$, let $u \in A$ be the unitary in the polar decomposition $x = u|x|$. Then,*

$$\|I - u\| \leq \sqrt{2}\|I - x\|.$$

- (2) *Let $p \in A$ be a projection and $a \in A$ a self-adjoint operator. Suppose that $\delta := \|a - p\| < 1/2$. Then $q := \chi_{[1-\delta, 1+\delta]}(a)$ is a projection in $C^*(a, I)$ satisfying*

$$\|q - p\| \leq 2\|a - p\| < 1.$$

- (3) *Let $p, q \in A$ be projections with $\|p - q\| < 1$. Then there exists a unitary $w \in A$ such that*

$$wpw^* = q \quad \text{and} \quad \|I - w\| \leq \sqrt{2}\|p - q\|.$$

In the paper, we consider metric between maps restricted to finite sets. The following are introduced by [9].

Definition 2.8. Let A and B be C^* -algebras and let $\phi_1, \phi_2: A \rightarrow B$ be maps. Given a subset $X \subseteq A$ and $\varepsilon > 0$, write $\phi_1 \approx_{X, \varepsilon} \phi_2$ if

$$\|\phi_1(x) - \phi_2(x)\| \leq \varepsilon, \quad x \in X.$$

Definition 2.9. Let A and B be C^* -algebras, X a subset of A and $\varepsilon > 0$. Given a completely positive contractive map (cpc map) $\phi: A \rightarrow B$, we call ϕ is an (X, ε) -approximate $*$ -homomorphism if it satisfies

$$\|\phi(x)\phi(x^*) - \phi(xx^*)\| \leq \varepsilon, \quad x \in X \cup X^*.$$

We only consider pairs of the form (x, x^*) in the previous definition by the following proposition, which can be found as [17, Lemma 7.11].

Proposition 2.10. *Let A and B be C^* -algebras and $\phi: A \rightarrow B$ a cpc map. Then for $x, y \in A$,*

$$\|\phi(xy) - \phi(x)\phi(y)\| \leq \|\phi(xx^*) - \phi(x)\phi(x^*)\|^{1/2}\|y\|.$$

Definition 2.11. Let A and B be C^* -algebras and let C be a C^* -subalgebras of $A \cap B$. A map $\phi: A \rightarrow B$ is C -fixed if $\phi|_C = \text{id}_C$.

Remark 2.12. Given a map $\phi: A \rightarrow B$ between C^* -algebras and a C^* -subalgebra C of $A \cap B$, if ϕ is C -fixed, then ϕ is a C -bimodule map, that is, for $x, z \in C$ and $y \in A$,

$$\phi(xyz) = x\phi(y)z.$$

The following lemma appears in [1, p.332]. We need the lemma in Lemma 3.7, 3.8 and 3.12.

Lemma 2.13. *Let $X \subseteq \mathbb{C}$ be a compact set and $\varepsilon, M > 0$. Then given a continuous function $f \in C(X)$, there exists $\eta > 0$ such that for any Hilbert space H , normal operator $s \in \mathbb{B}(H)$ with $\text{sp}(s) \subseteq X$ and $a \in \mathbb{B}(H)$ with $\|a\| \leq M$, the inequality $\|sa - as\| < \eta$ implies that $\|f(s)a - af(s)\| < \varepsilon$.*

Proof. Let p be a polynomial such that $\|f - p\| < \varepsilon/(4M)$, where this norm is the supremum norm of $C(X)$. Let p have the form $p(t) = c_0 + c_1t + \cdots + c_nt^n$. Define

$$\eta := \frac{\varepsilon}{2} \left(\sum_{k=1}^n k|c_k| \right)^{-1}.$$

Let a Hilbert space H be given and let a normal operator $s \in \mathbb{B}(H)$ and $a \in \mathbb{B}(H)_1$ satisfy $\|sa - as\| < \eta$. Let D be the derivation $D(x) = xa - ax$. Since $D(s^{n+1}) = sD(s^n) - D(s)s^n$, $\|D(s^{n+1})\| \leq \|D(s^n)\| + \|D(s)\|$. Hence, $\|D(s^n)\| \leq n\|D(s)\|$. Therefore,

$$\begin{aligned} \|f(s)a - af(s)\| &\leq \|f(s)a - p(s)a\| + \|D(p(s))\| + \|ap(s) - af(s)\| \\ &\leq 2\|f - p\|\|a\| + \sum_{k=1}^n k|c_k|\|D(s)\| < \varepsilon, \end{aligned}$$

and the lemma follows. \square

The next lemma appears in the proof of [9, Lemma 3.7].

Lemma 2.14. *Let H be a Hilbert space. Then for any $\mu_0 > 0$, there exists $\mu > 0$ with the following property: given a finite set $S \subseteq H_1$ and a self-adjoint operator $h \in \mathbb{B}(H)_1$, there exists a finite set $S' \subseteq H_1$ such that for any self-adjoint operator $k \in \mathbb{B}(H)_1$, if*

$$\|(h - k)\xi'\| < \mu, \quad \xi' \in S',$$

then

$$\|(e^{i\pi h} - e^{i\pi k})\xi\| < \mu_0 \quad \text{and} \quad \|(e^{i\pi h} - e^{i\pi k})^*\xi\| < \mu_0, \quad \xi \in S.$$

Proof. There exists a polynomial $p(t) = \sum_{j=0}^r \lambda_j t^j$ such that

$$|p(t) - e^{i\pi t}| < \frac{\mu_0}{3}, \quad -1 \leq t \leq 1. \quad (1)$$

Let

$$\mu := \frac{\mu_0}{3r \sum_{j=0}^r |\lambda_j|}. \quad (2)$$

Given a finite set $S \subseteq H_1$ and a self-adjoint operator $h \in \mathbb{B}(H)_1$, define

$$S' := \{h^m \xi : \xi \in S, m \leq r-1\}.$$

Let $k \in \mathbb{B}(H)_1$ be a self-adjoint operator with

$$\|(h - k)\xi'\| < \mu, \quad \xi' \in S'. \quad (3)$$

For any $\xi \in S$ and $0 \leq j \leq r$, by (3),

$$\begin{aligned} \|(h^j - k^j)\xi\| &\leq \|(h^j - kh^{j-1})\xi\| + \|(kh^{j-1} - k^2h^{j-2})\xi\| + \cdots + \|(k^{j-1}h - k^j)\xi\| \\ &\leq \|(h - k)h^{j-1}\xi\| + \|k(h - k)h^{j-2}\xi\| + \cdots + \|k^{j-1}(h - k)\xi\| \\ &\leq \sum_{m=0}^{j-1} \|(h - k)h^m\xi\| < r\mu. \end{aligned}$$

Thus, for $\xi \in S$,

$$\|(p(h) - p(k))\xi\| \leq \sum_{j=0}^r |\lambda_j| \|(h^j - k^j)\xi\| \leq \sum_{j=0}^r |\lambda_j| r\mu = \frac{\mu_0}{3},$$

by (2).

$$\begin{aligned} \|(e^{i\pi h} - e^{i\pi k})\xi\| &\leq \|(e^{i\pi h} - p(h))\xi\| + \|(p(h) - p(k))\xi\| + \|(p(k) - e^{i\pi k})\xi\| \\ &\leq \frac{\mu_0}{3} + \frac{\mu_0}{3} + \frac{\mu_0}{3} = \mu_0, \end{aligned}$$

by (1). Similarly, we have $\|(e^{i\pi h} - e^{i\pi k})^* \xi\| < \mu_0$. \square

3. CROSSED PRODUCT-LIKE INCLUSIONS AND APPROXIMATE AVERAGING

In this section, we introduce the crossed product-like inclusions of C^* -algebras. Moreover, we use the Følner condition of discrete amenable groups to modify the averaging results in [9, Section 3]. In Theorem 3.10, we show our first main result: Theorem A.

Given an inclusion $A \subseteq B$ of C^* -algebras, we denote by $\mathcal{N}_B(A)$ the normalizer of A in B , that is, $\mathcal{N}_B(A) = \{u \in B^u : uAu^* = A\}$.

Definition 3.1. Let $A \subseteq B$ be a unital inclusion of C^* -algebras. Then we call the inclusion $A \subseteq B$ *crossed product-like* if there exists a discrete group U in $\mathcal{N}_B(A)$ such that $B = C^*(A, U)$.

Since U is in $\mathcal{N}_B(A)$, $B = C^*(A, U)$ is the norm closure of $\text{span}\{au : a \in A, u \in U\}$. Throughout this paper, we only consider crossed product-like inclusions are by discrete *amenable* groups.

Remark 3.2. For any $x \in B$ and $\varepsilon > 0$, there exist $\{a_1, \dots, a_N\} \subseteq A_1$ and $\{u_1, \dots, u_N\} \subseteq U$ such that $\|x - \sum_{i=1}^N a_i u_i\| < \varepsilon$. In fact, let K be a positive integer with $K \geq \max\{\|a_1\|, \dots, \|a_N\|\}$. Define

$$a'_{(i-1)K+j} := \frac{1}{K} a_i, \quad i = 1, 2, \dots, N, j = 1, 2, \dots, K.$$

Then $a'_k \in A_1$ and

$$\sum_{i=1}^N a_i u_i = \sum_{j=1}^K \sum_{i=1}^N a'_{(i-1)K+j} u_i.$$

Example 3.3. Let G be a discrete amenable group acting on a C^* -algebra A . Then an inclusion $A \subseteq A \rtimes G$ is crossed product-like by $\{\lambda_g\}_{g \in G}$.

Example 3.4. Let (A, G, α, σ) be a twisted C^* -dynamical system and let $A \rtimes_{\alpha, r}^\sigma G$ be the reduced twisted crossed product. Then an inclusion $A \subseteq A \rtimes_{\alpha, r}^\sigma G$ is crossed product-like by $\{\lambda_\sigma(g)\}_{g \in G}$.

Example 3.5. Let $A \subseteq B$ be a crossed product-like inclusion of C^* -algebras by U . Then for a unital C^* -algebra C , $A \otimes C \subseteq B \otimes C$ is a crossed product inclusion by $U \otimes I$.

Remark 3.6. If $CI \subseteq A$ is a crossed product-like inclusion of C^* -algebras by a discrete amenable group, then A is strongly amenable. Hence, the Cuntz algebras \mathcal{O}_n are nuclear but $CI \subseteq \mathcal{O}_n$ is not crossed product-like by discrete amenable groups.

In the next lemma, to get a point-norm version of [6, Lemma 3.3] we modify the argument of [9, Lemma 3.2] for crossed product-like inclusions by amenable groups.

Lemma 3.7. *Let $C \subseteq D$ be a unital inclusion of C*-algebras and let A, B be intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group U and $d(A, B) < \gamma < 1/4$. Then for any finite subset $X \subseteq A_1$ and $\varepsilon > 0$, there exists a unital C -fixed (X, ε) -approximate *-homomorphism $\phi: A \rightarrow B$ such that*

$$\|\phi - \text{id}_A\| \leq (8\sqrt{2} + 2)\gamma.$$

Proof. Let a finite set $X \subseteq A_1$ and $0 < \varepsilon < 1$ be given. Let D act on a Hilbert space H . By Stinespring's theorem, we can find a Hilbert space $K \supseteq H$ and a unital *-homomorphism $\pi: D \rightarrow \mathbb{B}(K)$ such that

$$E(d) = P_H \pi(d)|_H, \quad d \in D,$$

because $E: D \rightarrow B$ is a unital cpc map. Furthermore, $P_H \in \pi(B)'$, since E is a B -fixed map. By Lemma 2.13, there exists $\eta > 0$ such that for any self-adjoint operator $t \in \mathbb{B}(K)$ with $\text{sp}(t) \subseteq [0, 2\gamma] \cup [1 - 2\gamma, 1]$ and $x \in \mathbb{B}(K)$ with $\|x\| \leq 2$, the inequality $\|xt - tx\| < \eta$ implies $\|xp - px\| < \varepsilon^2/18$, where p is the spectral projection of t for $[1 - 2\gamma, 1]$. There exist $\{u_1, \dots, u_N\} \subseteq U$ and $\{c_i^{(x)} : 1 \leq i \leq N, x \in X\} \subseteq C_1$ such that

$$\left\| x - \sum_{i=1}^N c_i^{(x)} u_i \right\| < \frac{\varepsilon}{3}, \quad x \in X.$$

Let $\tilde{x} := \sum_{i=1}^N c_i^{(x)} u_i$ for $x \in X$. Then $\|\tilde{x}\| \leq \|x\| + \varepsilon < 2$. Since U is amenable, we may choose a finite subset $F \subseteq U$ satisfying

$$\frac{|u_i F \triangle F|}{|F|} < \frac{\eta}{N}, \quad 1 \leq i \leq N.$$

Define

$$t := \frac{1}{|F|} \sum_{v \in F} \pi(v) P_H \pi(v^*) \in \mathbb{B}(K).$$

Since $U \subseteq \mathcal{N}_A(C)$ and $P_H \in \pi(C)'$, we have $t \in \pi(C)'$. For any $x \in X$,

$$\begin{aligned} \pi(\tilde{x})t &= \sum_{i=1}^N \pi(c_i^{(x)} u_i) \frac{1}{|F|} \sum_{v \in F} \pi(v) P_H \pi(v^*) \\ &= \frac{1}{|F|} \sum_{i=1}^N \sum_{v \in F} \pi(c_i^{(x)} u_i v) P_H \pi(v^*) \\ &= \frac{1}{|F|} \sum_{i=1}^N \sum_{\tilde{v} \in u_i F} \pi(c_i^{(x)} \tilde{v}) P_H \pi(\tilde{v}^* u_i) \end{aligned}$$

and

$$\begin{aligned} t\pi(\tilde{x}) &= \frac{1}{|F|} \sum_{v \in F} \pi(v) P_H \pi(v^*) \sum_{i=1}^N \pi(c_i^{(x)} u_i) \\ &= \frac{1}{|F|} \sum_{i=1}^N \sum_{v \in F} \pi(c_i^{(x)} v) P_H \pi(v^* u_i). \end{aligned}$$

Therefore,

$$\|\pi(\tilde{x})t - t\pi(\tilde{x})\| \leq \sum_{i=1}^N \frac{|u_i F \triangle F|}{|F|} < \eta, \quad x \in X. \quad (4)$$

For $v \in F$, there exists $v' \in B_1$ such that $\|v - v'\| < \gamma$. Since $P_H \in \pi(B)'$, we have

$$\|\pi(v)P_H - P_H\pi(v)\| \leq \|\pi(v)P_H - \pi(v')P_H\| + \|P_H\pi(v') - P_H\pi(v)\| \leq 2\gamma.$$

Thus, $\text{sp}(t) \subseteq [0, 2\gamma] \cup [1 - 2\gamma, 1]$, since

$$\begin{aligned} \|t - P_H\| &= \left\| \frac{1}{|F|} \sum_{v \in F} \pi(v) P_H \pi(v^*) - \frac{1}{|F|} \sum_{v \in F} P_H \pi(v) \pi(v^*) \right\| \\ &\leq \frac{1}{|F|} \sum_{v \in F} \|\pi(v)P_H - P_H\pi(v)\| \|\pi(v^*)\| \leq 2\gamma. \end{aligned}$$

Let $q = \chi_{[1-2\gamma, 1]}(t) \in C^*(t, I_K)$. By (4),

$$\|\pi(\tilde{x})q - q\pi(\tilde{x})\| < \frac{\varepsilon^2}{18}, \quad x \in X. \quad (5)$$

Since $\|q - P_H\| \leq 2\|t - P_H\| < 1$, there exists a unitary $w \in C^*(t, P_H, I_K)$ such that $wP_Hw^* = q$ and $\|w - I_K\| \leq \sqrt{2}\|q - P_H\|$. Define $\phi: A \rightarrow \mathbb{B}(K)$ by

$$\phi(a) = P_Hw^*\pi(a)w|_H, \quad a \in A.$$

Since $w \in C^*(t, P_H, I_K) \subseteq C^*(\pi(A), P_H)$ and $P_H\pi(A)|_H = \text{ran}(E) \subset B$, the range of ϕ is contained in B . Furthermore, $\phi|_C = \text{id}_C$ because $w \in C^*(t, P_H, I_K) \subseteq \pi(C)'$.

For $x \in X \cup X^*$, by using $P_Hw^* = P_Hw^*q$ and (10),

$$\begin{aligned} \|\phi(\tilde{x}\tilde{x}^*) - \phi(\tilde{x})\phi(\tilde{x}^*)\| &= \|P_Hw^*\pi(\tilde{x}\tilde{x}^*)wP_H - P_Hw^*\pi(\tilde{x})wP_Hw^*\pi(\tilde{x}^*)wP_H\| \\ &= \|P_Hw^*q\pi(\tilde{x}\tilde{x}^*)wP_H - P_Hw^*\pi(\tilde{x})q\pi(\tilde{x}^*)wP_H\| \\ &\leq \|q\pi(\tilde{x}) - \pi(\tilde{x})q\| \|\pi(\tilde{x}^*)\| < \frac{\varepsilon^2}{9}. \end{aligned} \quad (6)$$

Therefore, by (6) and Proposition 2.10,

$$\begin{aligned} &\|\phi(xx^*) - \phi(x)\phi(x^*)\| \\ &\leq \|\phi(xx^*) - \phi(x\tilde{x}^*)\| + \|\phi(x\tilde{x}^*) - \phi(x)\phi(\tilde{x}^*)\| + \|\phi(x)\phi(\tilde{x}) - \phi(x)\phi(x^*)\| \\ &\leq \|xx^* - x\tilde{x}^*\| + \|\phi(\tilde{x}\tilde{x}^*) - \phi(\tilde{x})\phi(\tilde{x}^*)\|^{1/2}\|x\| + \|\phi(x)\| \|\phi(\tilde{x}) - \phi(x^*)\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

For $a \in A_1$, we have

$$\begin{aligned} \|\phi(a) - a\| &\leq \|\phi(a) - E(a)\| + \|E(a) - a\| \\ &\leq \|P_H w^* \pi(a) w P_H - P_H \pi(a) P_H\| + 2d(A, B) \\ &\leq 2\|w - I_K\| + 2d(A, B) \leq (8\sqrt{2} + 2)\gamma, \end{aligned}$$

and the lemma follows. \square

The next lemma is a version of [9, Lemma 3.4] for crossed product-like inclusions by amenable groups.

Lemma 3.8. *Let A, B and C be C*-algebras with a common unit. Suppose that $C \subseteq A \cap B$ and $C \subseteq A$ is crossed product-like by a discrete amenable group U . Then for any finite set $X \subseteq A_1$ and $\varepsilon > 0$, there exist a finite set $Y \subseteq A_1$ and $\delta > 0$ with the following property: Given $\gamma < 1/10$ and two unital C -fixed (Y, δ) -approximation *-homomorphisms $\phi_1, \phi_2: A \rightarrow B$ with $\phi_1 \approx_{Y, \gamma} \phi_2$, there exists a unitary $u \in C' \cap B$ such that*

$$\phi_1 \approx_{X, \varepsilon} \text{Ad}(u) \circ \phi_2 \quad \text{and} \quad \|u - I\| \leq \sqrt{2}(\gamma + \delta).$$

Proof. Let a finite set $X \subseteq A_1$ and $0 < \varepsilon < 1$ be given. There exist $\{u_1, \dots, u_N\} \subseteq U$ and $\{c_i^{(x)} : 1 \leq i \leq N, x \in X\} \subseteq C_1$ such that

$$\left\| x - \sum_{i=1}^N c_i^{(x)} u_i \right\| < \frac{\varepsilon}{3}, \quad x \in X.$$

Let $\tilde{x} := \sum_{i=1}^N c_i^{(x)} u_i^{(x)}$ for $x \in X$. Then $\|\tilde{x}\| \leq 1 + \varepsilon/3 < 2$. By Lemma 2.13, there exists $\eta > 0$ such that for any $s \in B_1$ and $a \in B$ with $\|a\| \leq 2$, the inequality $\|ss^*a - ass^*\| < \eta$ implies $\|s|a - a|s\| < \varepsilon/12$. Let

$$0 < \delta < \min \left\{ \left(\frac{\varepsilon}{60} \right)^2, \frac{\eta^2}{100} \right\}.$$

There exists a finite set $Y \subseteq U$ such that

$$\frac{|uY \triangle Y|}{|Y|} < \frac{\delta}{N}, \quad u \in \{u_i, u_i^* : 1 \leq i \leq N\}.$$

Let $\gamma < 1/10$ and $\phi_1, \phi_2: A \rightarrow B$ be C -fixed (Y, δ) -approximation *-homomorphisms with $\phi_1 \approx_{Y, \gamma} \phi_2$. Define

$$s := \frac{1}{|Y|} \sum_{v \in Y} \phi_1(v) \phi_2(v^*).$$

Since ϕ_1 and ϕ_2 are C -fixed maps and for $u \in U$, $uC u^* = C$, we have $s \in C' \cap B$. By Proposition 2.10, for $x \in X$ and $v \in Y$,

$$\|\phi_1(\tilde{x}v) - \phi_1(\tilde{x})\phi_1(v)\| \leq \|\phi_1(vv^*) - \phi_1(v)\phi_1(v^*)\|^{1/2} \|\tilde{x}\| \leq 2\sqrt{\delta}, \quad (7)$$

$$\|\phi_2(v^*\tilde{x}) - \phi_2(v^*)\phi_2(\tilde{x})\| \leq \|\phi_2(v^*v) - \phi_2(v^*)\phi_2(v)\|^{1/2} \|\tilde{x}\| \leq 2\sqrt{\delta}. \quad (8)$$

Furthermore,

$$\begin{aligned} \frac{1}{|Y|} \sum_{v \in Y} \phi_1(\tilde{x}v) \phi_2(v^*) &= \frac{1}{|Y|} \sum_{v \in Y} \sum_{i=1}^N \phi_1(c_i^{(x)} u_i v) \phi_2(v^*) \\ &= \frac{1}{|Y|} \sum_{i=1}^N \sum_{v \in u_i Y} \phi_1(c_i^{(x)} v) \phi_2(v^* u_i) \end{aligned} \quad (9)$$

and

$$\begin{aligned} \frac{1}{|Y|} \sum_{v \in Y} \phi_1(v) \phi_2(v^* \tilde{x}) &= \frac{1}{|Y|} \sum_{v \in Y} \sum_{i=1}^N \phi_1(v) \phi_2(v^* c_i^{(x)} u_i) \\ &= \frac{1}{|Y|} \sum_{i=1}^N \sum_{v \in Y} \phi_1(c_i^{(x)} v) \phi_2(v^* u_i) \end{aligned} \quad (10)$$

By (9), (10) and the choice of Y , for $x \in X$,

$$\left\| \frac{1}{|Y|} \sum_{v \in Y} \phi_1(\tilde{x}v) \phi_2(v^*) - \frac{1}{|Y|} \sum_{v \in Y} \phi_1(v) \phi_2(v^* \tilde{x}) \right\| < \sum_{i=1}^N \frac{|u_i^{(x)} Y \triangle Y|}{|Y|} < \delta. \quad (11)$$

Similarly, we have

$$\left\| \frac{1}{|Y|} \sum_{v \in Y} \phi_1(\tilde{x}^* v) \phi_2(v^*) - \frac{1}{|Y|} \sum_{v \in Y} \phi_1(v) \phi_2(v^* \tilde{x}^*) \right\| < \delta, \quad x \in X. \quad (12)$$

By (7), (8) and (11),

$$\|\phi_1(\tilde{x})s - s\phi_2(\tilde{x})\| \leq \delta + 4\sqrt{\delta} < 5\sqrt{\delta}, \quad x \in X \cup X^*. \quad (13)$$

By taking adjoints,

$$\|s^* \phi_1(\tilde{x}) - \phi_2(\tilde{x})s^*\| \leq 5\sqrt{\delta}, \quad x \in X \cup X^*.$$

Thus, for $x \in X \cup X^*$,

$$\begin{aligned} \|\phi_2(\tilde{x})s^*s - s^*s\phi_2(\tilde{x})\| &\leq \|\phi_2(\tilde{x})s^*s - s^*\phi_1(\tilde{x})s\| + \|s^*\phi_1(\tilde{x})s - s^*s\phi_2(\tilde{x})\| \\ &\leq \|\phi_2(\tilde{x})s^* - s^*\phi_1(\tilde{x})\| \|s\| + \|s^*\| \|\phi_1(\tilde{x})s - s\phi_2(\tilde{x})\| \\ &\leq 10\sqrt{\delta} < \eta. \end{aligned}$$

By the choice of η and (14),

$$\|\phi_2(\tilde{x})|s| - |s|\phi_2(\tilde{x})\| < \frac{\varepsilon}{12}, \quad x \in X \cup X^*. \quad (14)$$

Since ϕ_1 is a (Y, δ) -approximation $*$ -homomorphism and $\phi_1 \approx_{Y, \gamma} \phi_2$, we have

$$\begin{aligned} \|s - I\| &= \left\| \frac{1}{|Y|} \sum_{v \in Y} \phi_1(v) \phi_2(v^*) - \frac{1}{|Y|} \sum_{v \in Y} \phi_1(vv^*) \right\| \\ &\leq \frac{1}{|Y|} \sum_{v \in Y} \|\phi_1(v) \phi_2(v^*) - \phi_1(v) \phi_1(v^*)\| + \frac{1}{|Y|} \sum_{v \in Y} \|\phi_1(v) \phi_1(v^*) - \phi_1(vv^*)\| \\ &\leq \gamma + \delta < 1. \end{aligned} \quad (15)$$

Since this inequality gives invertibility of s , the unitary u in the polar decomposition $s = u|s|$ lies in $C^*(s, I) \subseteq C' \cap B$ and satisfies $\|u - I\| \leq \sqrt{2}(\gamma + \delta)$. Then, by (15),

$$\||s| - I\| \leq \|u^*s - I\| \leq \|s - I\| + \|I - u\| \leq (1 + \sqrt{2})(\gamma + \delta) < \frac{1}{2}.$$

Hence, $\||s|^{-1}\| \leq 2$ so,

$$\begin{aligned} \|\phi_1(\tilde{x}) - u\phi_2(\tilde{x})u^*\| &= \|\phi_1(\tilde{x})u - u\phi_2(\tilde{x})\| \\ &\leq \|\phi_1(\tilde{x})u|s| - u\phi_2(\tilde{x})|s|\| \||s|^{-1}\| \\ &\leq 2\|\phi_1(\tilde{x})u|s| - u\phi_2(\tilde{x})|s|\| \\ &\leq 2\|\phi_1(\tilde{x})s - s\phi_2(\tilde{x})\| + 2\|s\phi_2(\tilde{x}) - u\phi_2(\tilde{x})|s|\| \\ &\leq 10\sqrt{\delta} + 2\||s|\phi_2(\tilde{x}) - \phi_2(\tilde{x})|s|\| \\ &\leq 10\sqrt{\delta} + \frac{\varepsilon}{6} < \frac{\varepsilon}{3}, \end{aligned} \tag{16}$$

for $x \in X$, by (13), (14) and (15). For $x \in X$, by (16),

$$\begin{aligned} \|\phi_1(x) - u\phi_2(x)u^*\| &\leq \|\phi_1(x) - \phi_1(\tilde{x})\| + \|\phi_1(\tilde{x}) - u\phi_2(\tilde{x})u^*\| + \|u\phi_2(\tilde{x})u^* - u\phi_2(x)u^*\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, $\phi_1 \approx_{X, \varepsilon} \text{Ad}(u) \circ \phi_2$. \square

Remark 3.9. Let a pair (Y, δ) hold Lemma 3.8. Then for any finite set $Y' \supseteq Y$ and constant $\delta' < \delta$, a pair (Y', δ') holds Lemma 3.8.

By Lemma 3.7 and 3.8, we can show Theorem A.

Theorem 3.10. *Let $C \subseteq D$ be a unital irreducible inclusion of C*-algebras acting on a separable Hilbert space H . Let A and B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group. If $d(A, B) < 140^{-1}$, then $A = B$.*

Proof. Let $a \in A_1, \varepsilon > 0$ and $d(A, B) < \gamma < 140^{-1}$ be given. By Lemma 3.8, there exist a finite subset $Y \subseteq A_1$ and $\delta > 0$ with the following property: Given $\gamma' < 1/10$ and two unital C -fixed (Y, δ) -approximate *-homomorphisms $\phi_1, \phi_2: A \rightarrow D$ with $\phi_1 \approx_{Y, \gamma'} \phi_2$, there exists a unitary $u \in C' \cap D$ such that

$$\|\phi_1(a) - (\text{Ad}(u) \circ \phi_2)(a)\| \leq \varepsilon.$$

By Lemma 3.7, there exists a unital C -fixed (Y, δ) -approximate *-homomorphism $\phi: A \rightarrow B$ such that $\|\phi - \text{id}_A\| \leq (8\sqrt{2} + 2)\gamma$. Then there exists a unitary $u \in C' \cap D$ such that

$$\|\phi(a) - (\text{Ad}(u))(a)\| \leq \varepsilon$$

by the definition of Y and δ . Since $u \in C' \cap D = \mathbb{C}I$, we have $\|\phi(a) - a\| \leq \varepsilon$. Therefore, since $\phi(a) \in B$ and ε is arbitrary, $a \in B$, that is, $A \subseteq B$. Furthermore, by Lemma 2.4, the theorem follows. \square

In the following lemma, we modify [9, Lemma 3.6], which is a Kaplansky density style result for approximate commutants.

Lemma 3.11. *Let $C \subseteq A$ be a unital inclusion of non-degenerate C^* -algebras in $\mathbb{B}(H)$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group U . Then for any finite set $X \subseteq A_1$ and $\varepsilon, \mu > 0$, there exist a finite set $Y \subseteq A_1$ and $\delta > 0$ with the following property: Given a finite set $S \subseteq H_1$ and a self-adjoint operator $m \in \overline{C' \cap A_1}^w$ with*

$$\|my - ym\| \leq \delta, \quad y \in Y, \quad (17)$$

there exists a self-adjoint operator $a \in C' \cap A_1$ such that $\|a\| \leq \|m\|$,

$$\|ax - xa\| < \varepsilon, \quad x \in X, \quad (18)$$

and

$$\|(a - m)\xi\| < \mu \quad \text{and} \quad \|(a - m)^*\xi\| < \mu, \quad \xi \in S. \quad (19)$$

Proof. Let a finite set $X \subseteq A_1$ and $\varepsilon, \mu > 0$ be given. There exist $\{u_1, \dots, u_N\} \subseteq U$ and $\{c_i^{(x)} : 1 \leq i \leq N, x \in X\} \subseteq C_1$ such that

$$\left\| x - \sum_{i=1}^N c_i^{(x)} u_i \right\| < \frac{\varepsilon}{3}, \quad x \in X.$$

Let $\tilde{x} := \sum_{i=1}^N c_i^{(x)} u_i$ for $x \in X$. Since U is amenable, there exists a finite set $F \subseteq U$ such that

$$\frac{|u_i F \triangle F|}{|F|} < \frac{\varepsilon}{3N}, \quad 1 \leq i \leq N. \quad (20)$$

Define $Y := F \cup F^*$ and $\delta := \mu/2$.

Let S be a finite set in H_1 and $m \in \overline{C' \cap A_1}^w$ be a self-adjoint operator with

$$\|my - ym\| < \delta, \quad y \in Y. \quad (21)$$

By Kaplansky's density theorem, there exists a self-adjoint operator $a_0 \in C' \cap A_1$ such that $\|a_0\| \leq \|m\|$,

$$\|(a_0 - m)v^*\xi\| < \mu \quad \text{and} \quad \|(a_0 - m)^*v\xi\| < \mu, \quad v \in Y, \quad \xi \in S. \quad (22)$$

Define

$$a := \frac{1}{|F|} \sum_{v \in F} v a_0 v^*.$$

Then, $\|a\| \leq \|a_0\| \leq \|m\|$.

For any $x \in X$,

$$\begin{aligned} \|\tilde{x}a - a\tilde{x}\| &= \left\| \frac{1}{|F|} \sum_{i=1}^N \sum_{v \in F} c_i^{(x)} u_i v a_0 v^* - \frac{1}{|F|} \sum_{i=1}^N \sum_{v \in F} c_i^{(x)} v a_0 v^* u_i \right\| \\ &= \left\| \frac{1}{|F|} \sum_{i=1}^N c_i^{(x)} \left(\sum_{\tilde{v} \in u_i F} \tilde{v} a_0 \tilde{v}^* u_i - \sum_{v \in F} v a_0 v^* u_i \right) \right\| \\ &\leq \sum_{i=1}^N \frac{|u_i F \triangle F|}{|F|} < \frac{\varepsilon}{3}, \end{aligned} \quad (23)$$

by (20). For $x \in X$, since $\|x - \tilde{x}\| < \varepsilon/3$,

$$\begin{aligned} \|xa - ax\| &\leq \|xa - \tilde{x}a\| + \|\tilde{x}a - a\tilde{x}\| + \|a\tilde{x} - xa\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

by (23).

For $\xi \in S$, by (21) and (22),

$$\begin{aligned} &\|(a - m)\xi\| \\ &\leq \left\| \left(\frac{1}{|F|} \sum_{v \in F} va_0v^* - \frac{1}{|F|} \sum_{v \in F} vmv^* \right) \xi \right\| + \left\| \left(\frac{1}{|F|} \sum_{v \in F} vmv^* - \frac{1}{|F|} \sum_{v \in F} vv^*m \right) \xi \right\| \\ &\leq \max_{v \in F} \|(a_0 - m)v^*\xi\| + \max_{v \in F} \|mv^* - v^*m\| \\ &\leq \frac{\mu}{2} + \delta = \mu. \end{aligned}$$

Similarly, for $\xi \in S$,

$$\|(a - m)^*\xi\| \leq \max_{v \in F} \|(a_0 - m)^*v\xi\| + \max_{v \in F} \|mv - vm\| \leq \frac{\mu}{2} + \delta = \mu,$$

and the lemma follows. \square

By Lemma 2.14 and 3.11, we obtain the following version of Lemma 3.11 for unitary operators. We need the next lemma in Section 6.

Lemma 3.12. *Let $C \subseteq A$ be a unital inclusion of non-degenerate C*-algebras in $\mathbb{B}(H)$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group U . Then for any finite set $X \subseteq A_1$, $\varepsilon_0, \mu_0 > 0$ and $0 < \alpha < 2$, there exist a finite set $Y \subseteq A_1$ and $\delta_0 > 0$ with the following property: Given a finite set $S \subseteq H_1$ and a unitary $u \in \overline{C' \cap A}^w$ with $\|u - I_H\| \leq \alpha$ and*

$$\|uy - yu\| \leq \delta_0, \quad y \in Y, \quad (24)$$

there exists a unitary $v \in C' \cap A$ such that $\|v - I_H\| \leq \|u - I_H\|$,

$$\|vx - xv\| < \varepsilon_0, \quad x \in X, \quad (25)$$

and

$$\|(v - u)\xi\| < \mu_0 \quad \text{and} \quad \|(v - u)^*\xi\| < \mu_0, \quad \xi \in S. \quad (26)$$

Proof. Let a finite set $X \subseteq A_1$, $\varepsilon_0, \mu_0 > 0$ and $0 < \alpha < 2$ be given. There exists $0 < c < \pi$ such that $|1 - e^{i\pi\theta}| \leq \alpha$ if and only if $\theta \in [-c, c]$ modulo 2π . By Lemma 2.13, there exists $\varepsilon > 0$ such that given a self-adjoint operator $k \in \mathbb{B}(H)_1$ and $a \in \mathbb{B}(H)_1$, if $\|ak - ka\| < \varepsilon$, then $\|ae^{i\pi k} - e^{i\pi k}a\| < \varepsilon_0$.

By Lemma 2.14, there exists $\mu > 0$ with the following property: Given a finite set $S \subseteq H_1$ and a self-adjoint operator $k \in \mathbb{B}(H)_1$, there exists a finite set $S' \subseteq H_1$ such that for a self-adjoint operator $k \in \mathbb{B}(H)_1$, if

$$\|(h - k)\xi'\| < \mu_0, \quad \xi' \in S',$$

then

$$\|(e^{i\pi h} - e^{i\pi k})\xi\| < \mu \quad \text{and} \quad \|(e^{i\pi h} - e^{i\pi k})^*\xi\| < \mu, \quad \xi \in S.$$

By Lemma 3.11, there exist a finite set $Y \subseteq A_1$ and $\delta > 0$ with the following property: For any finite set $S \subseteq H_1$ and self-adjoint operator $m \in \overline{C' \cap A_1}^w$ with

$$\|my - ym\| < \delta, \quad y \in Y,$$

there exists a self-adjoint operator $a \in C' \cap A_1$ such that $\|a\| \leq \|m\|$,

$$\begin{aligned} \|ax - xa\| &< \varepsilon, \quad x \in X, \\ \|(a - m)\xi\| &< \mu \quad \text{and} \quad \|(a - m)^*\xi\| < \mu, \quad \xi \in S. \end{aligned}$$

By Lemma 2.13, there exists $\delta_0 > 0$ such that for any $y \in \mathbb{B}(H)$ and unitary $u \in \mathbb{B}(H)$ with $\|u - I_H\| \leq \alpha$, if $\|uy - yu\| \leq \delta_0$, then

$$\left\| \frac{\log u}{\pi} y - y \frac{\log u}{\pi} \right\| \leq \delta.$$

Given a finite set $S \subset H_1$ and a unitary $u \in \overline{C' \cap A}^w$ with $\|u - I_H\| \leq \alpha$ and

$$\|uy - yu\| \leq \delta_0, \quad y \in Y.$$

Let

$$h := -i \frac{\log u}{\pi} \in C' \cap M.$$

By the definition of δ_0 ,

$$\|hy - yh\| \leq \delta, \quad y \in Y.$$

By the definition of μ , there exists a finite set $S' \subseteq H_1$ such that for any self-adjoint operator $k \in \mathbb{B}(H)_1$, if

$$\|(h - k)\xi'\| < \mu_0, \quad \xi' \in S',$$

then

$$\|(e^{i\pi h} - e^{i\pi k})\xi\| < \mu \quad \text{and} \quad \|(e^{i\pi h} - e^{i\pi k})^*\xi\| < \mu, \quad \xi \in S.$$

By the definitions of Y and δ , there exists a self-adjoint operator $k \in C' \cap A_1$ such that $\|k\| \leq \|h\|$,

$$\begin{aligned} \|kx - xk\| &< \varepsilon, \quad x \in X, \\ \|(h - k)\xi'\| &< \mu \quad \text{and} \quad \|(h - k)^*\xi'\| < \mu, \quad \xi' \in S'. \end{aligned}$$

Define $v := e^{i\pi k}$. Then, we have $\|v - I_H\| \leq \|e^{i\pi h} - I_H\| = \|u - I_H\|$.

By the definition of ε and S' , we have

$$\|vx - xv\| < \varepsilon_0, \quad x \in X$$

and

$$\|(v - u)\xi\| < \mu_0 \quad \text{and} \quad \|(v - u)^*\xi\| < \mu_0, \quad \xi \in S.$$

Hence the lemma is proved. \square

4. ISOMORPHISMS

In this section, we show Theorem B. Given a unital inclusion $C \subseteq D$ of C*-algebras and intermediate C*-subalgebras A, B for this inclusion with a conditional expectation from D onto B , if $A = C \rtimes G$, where G is a discrete amenable group, and if A and B are sufficiently close, then A must be *-isomorphic to B . To do this, we modify [9, Lemma 4.1] in the next lemma. The approximation approach of [9, Lemma 4.1] inspired by the intertwining arguments of [8, Theorem 6.1].

Lemma 4.1. *Let $C \subseteq D$ be a unital inclusion of C*-algebras and let A, B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Let $\{a_n\}_{n=0}^\infty$ be a dense subset in A_1 with $a_0 = 0$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-3}$. Put $\eta := (8\sqrt{2} + 2)\gamma$. Then for any finite set $X \subseteq A_1$, there exist finite subsets $\{X_n\}_{n=0}^\infty, \{Y_n\}_{n=0}^\infty \subseteq A_1$, positive constants $\{\delta_n\}_{n=0}^\infty$, C -fixed cpc maps $\{\theta_n: A \rightarrow B\}_{n=0}^\infty$ and unitaries $\{u_n\}_{n=1}^\infty \subseteq C' \cap B$ with the following conditions:*

- (a) *For $n \geq 0$, $\delta_n < \min\{2^{-n}, \gamma\}$, $a_n \in X_n \subseteq X_{n+1}$ and $X \subseteq X_1$;*
- (b) *For $n \geq 0$ and two unital C -fixed (Y_n, δ_n) -approximation *-homomorphisms $\phi_1, \phi_2: A \rightarrow B$ with $\phi_1 \approx_{Y_n, 2\eta} \phi_2$, there exists a unitary $u \in C' \cap B$ such that $\text{Ad}(u) \circ \phi_1 \approx_{X_n, \gamma/2^n} \phi_2$ and $\|u - I\| \leq \sqrt{2}(2\eta + \delta_n)$;*
- (c) *For $n \geq 0$, $X_n \subseteq Y_n$;*
- (d) *For $n \geq 0$, θ_n is a (Y_n, δ_n) -approximation *-homomorphism with $\|\theta_n - \text{id}_A\| \leq \eta$;*
- (e) *For $n \geq 1$, $\text{Ad}(u_n) \circ \theta_n \approx_{X_{n-1}, \gamma/2^{n-1}} \theta_{n-1}$ and $\|u_n - I\| \leq \sqrt{2}(2\eta + \delta_{n-1})$.*

Proof. We prove this lemma by the induction. Let a finite subset $X \subseteq A_1$ be given. Let $X_0 = \{0\} = \{a_0\} = Y_0$, $\delta = 1$ and $\theta := E|_A: A \rightarrow B$.

Suppose that we can construct completely up to the n -th stage. We will write the condition (a) for n as $(a)_n$. Let $X_{n+1} := X_n \cup X \cup \{a_{n+1}\} \cup Y_n$. By Lemma 3.8, there exist a finite set $Y_{n+1} \subseteq A_1$ and $0 < \delta_{n+1} < \min\{\delta_n, 2^{-(n+1)}, \gamma\}$ satisfying condition $(b)_{n+1}$ and $X_{n+1} \subseteq Y_{n+1}$. By Lemma 3.7, there exists a unital C -fixed (Y_{n+1}, δ_{n+1}) -approximation *-homomorphism $\theta_{n+1}: A \rightarrow B$ such that $\|\theta_{n+1} - \text{id}_A\| \leq \eta$. Therefore, $X_{n+1}, Y_{n+1}, \delta_{n+1}$ and θ_{n+1} satisfy $(a)_{n+1}, (b)_{n+1}, (c)_{n+1}$ and $(d)_{n+1}$.

Since $Y_n \subseteq Y_{n+1}$ and $\delta_{n+1} < \delta_n$, θ_n and θ_{n+1} are unital C -fixed (Y_n, δ_n) -approximation *-homomorphisms with $\|\theta_n - \theta_{n+1}\| \leq 2\eta$. Thus, by $(b)_n$, there exists a unitary $u_{n+1} \in C' \cap B$ such that $\text{Ad}(u_{n+1}) \circ \theta_{n+1} \approx_{X_n, \gamma/2^n} \theta_n$ and $\|u_{n+1} - I\| \leq \sqrt{2}(2\eta + \delta_n)$. Then $(e)_{n+1}$ follows. \square

Proposition 4.2. *Let $C \subseteq D$ be a unital inclusion of C*-algebras and let A and B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-3}$. Then for any finite subset $X \subseteq A_1$, there exists a C -fixed surjective *-isomorphism $\alpha: A \rightarrow B$ such that*

$$\alpha \approx_{X, 15\gamma} \text{id}_A.$$

Proof. Let $\{a_n\}_{n=0}^\infty$ be a dense subset in A_1 with $a_0 = 0$. Put $\eta := (8\sqrt{2} + 2)\gamma$. By Lemma 4.1, we can construct $\{X_n\}_{n=0}^\infty, \{Y_n\}_{n=0}^\infty \subseteq A_1$, $\{\delta_n\}_{n=0}^\infty$, $\{\theta_n: A \rightarrow B\}_{n=0}^\infty$ and $\{u_n\}_{n=1}^\infty \subseteq C' \cap B$ which satisfy conditions (a)-(e) of that lemma. For any $n \geq 1$, define

$$\alpha_n := \text{Ad}(u_1 \cdots u_n) \circ \theta_n.$$

Fix $k \in \mathbb{N}$ and $x \in X_k$. For any $n \geq k$,

$$\begin{aligned} \|\alpha_{n+1}(x) - \alpha_n(x)\| &= \|(\text{Ad}(u_1 \cdots u_n) \circ \text{Ad}(u_{n+1}) \circ \theta_{n+1} - \text{Ad}(u_1 \cdots u_n) \circ \theta_n)(x)\| \\ &= \|(\text{Ad}(u_{n+1}) \circ \theta_{n+1} - \theta_n)(x)\| \leq \frac{\gamma}{2^n} \end{aligned} \quad (27)$$

For any $\varepsilon > 0$, there exists $N \geq k$ such that $\gamma/2^{N-1} < \varepsilon$. For any two natural numbers $m \geq n \geq N$, by (27),

$$\|\alpha_m(x) - \alpha_n(x)\| \leq \sum_{j=n}^{m-1} \|\alpha_{j+1}(x) - \alpha_j(x)\| \leq \sum_{j=n}^{m-1} \frac{\gamma}{2^j} < \frac{\gamma}{2^{N-1}} < \varepsilon.$$

Thus, $\{\alpha_n(x)\}$ is a Cauchy sequence. Since $\bigcup_{n=0}^{\infty} X_n$ is dense in A_1 , the sequence $\{\alpha_n\}$ converges in the point-norm topology to a C -fixed cpc map $\alpha: A \rightarrow B$. Moreover, α is a $*$ -homomorphism, since $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\bigcup_{n=0}^{\infty} Y_n$ is dense in A_1 .

For any $n \in \mathbb{N}$ and $x \in X_n$,

$$\|\alpha(x) - \alpha_n(x)\| \leq \sum_{j=n}^{\infty} \|\alpha_{j+1}(x) - \alpha_j(x)\| \leq \sum_{j=n}^{\infty} \frac{\gamma}{2^j} = \frac{\gamma}{2^{n-1}}. \quad (28)$$

Hence, by (e) in Lemma 4.1,

$$\begin{aligned} \|\alpha(x)\| &\geq \| \|\alpha(x) - \alpha_n(x)\| - \|\alpha_n(x)\| \| \\ &\geq \|\alpha_n(x)\| - \frac{\gamma}{2^{n-1}} \\ &= \|\theta_n(x)\| - \frac{\gamma}{2^{n-1}} \\ &\geq \| \|\theta_n(x) - x\| - \|x\| \| - \frac{\gamma}{2^{n-1}} \\ &\geq \|x\| - \eta - \frac{\gamma}{2^{n-1}}. \end{aligned} \quad (29)$$

Let $n \rightarrow \infty$ in (29). Then, for any x in the unit sphere of A , we have

$$\|\alpha(x)\| \geq 1 - \eta > 0, \quad (30)$$

by the density of $\bigcup_{n=0}^{\infty} X_n$ in A_1 . Therefore, α is an injective map.

For any $b \in B_1$ and $n \in \mathbb{N}$, there exists $x \in A_1$ such that $\|x - u_n^* \cdots u_1^* b u_1 \cdots u_n\| < \gamma$.

$$\begin{aligned} \|\alpha_n(x) - b_j\| &= \|u_1 \cdots u_n \theta_n(x) u_n^* \cdots u_1^* - b\| \\ &\leq \|\theta_n(x) - x\| + \|x - u_n^* \cdots u_1^* b u_1 \cdots u_n\| \\ &< \eta + \gamma < 1. \end{aligned}$$

Thus, $d(\alpha(A), B) < 1$, that is, α is a surjective map by Proposition 2.4.

For any $x \in X$,

$$\|\alpha(x) - x\| \leq \|\alpha(x) - \alpha_1(x)\| + \|\theta_1(x) - x\| \leq \gamma + \eta < 15\gamma.$$

by (29) and (e) in Lemma 4.1. □

Theorem 4.3. *Let $C \subseteq D$ be a unital inclusion of C^* -algebras and let A and B be separable intermediate C^* -subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-3}$.*

Then for any finite subset $X \subseteq A_1$ and finite set $Y \subseteq B_1$, there exists a C -fixed surjective $*$ -isomorphism $\alpha: A \rightarrow B$ such that

$$\alpha \approx_{X,15\gamma} \text{id}_A \text{ and } \alpha^{-1} \approx_{Y,17\gamma} \text{id}_B.$$

Proof. There exists a finite set $\tilde{X} \subseteq A_1$ such that $\tilde{X} \subset_\gamma Y$. By Proposition 4.2, there exists a C -fixed surjective $*$ -isomorphism $\alpha: A \rightarrow B$ such that

$$\alpha \approx_{X \cup \tilde{X}, 15\gamma} \text{id}_A.$$

Fix $y \in Y$. Since $\tilde{X} \subset_\gamma Y$, there exists $\tilde{x} \in \tilde{X}$ such that $\|\tilde{x} - y\| < \gamma$. Then, we have

$$\begin{aligned} \|\alpha^{-1}(y) - y\| &\leq \|\alpha^{-1}(y) - \tilde{x}\| + \|\tilde{x} - y\| \\ &\leq \|y - \alpha(\tilde{x})\| + \gamma \\ &\leq \|y - \tilde{x}\| + \|\tilde{x} - \alpha(\tilde{x})\| + \gamma \\ &\leq \gamma + 15\gamma + \gamma \leq 17\gamma. \end{aligned}$$

Therefore, $\alpha^{-1} \approx_{Y, 17\gamma} \text{id}_B$. □

5. CROSSED PRODUCT VON NEUMANN ALGEBRAS BY AMENABLE GROUPS

In Theorem 5.7, we now show that given a unital inclusion $N \subseteq M$ of von Neumann algebras and intermediate von Neumann subalgebras A, B for this inclusion with a normal conditional expectation from M onto B , if $A = N \rtimes G$, where G is a discrete amenable group, and if A and B are sufficiently close, then there exists a unitary $u \in N' \cap M$ such that $uAu^* = B$. This unitary can be chosen to be close to the identity.

Definition 5.1. Let $N \subseteq M$ be a unital inclusion of von Neumann algebras in $\mathbb{B}(H)$. Then we call the inclusion $N \subseteq M$ is *crossed product-like* if there exists a discrete group U in $\mathcal{N}_M(N)$ such that $M = (N \cup U)''$.

Example 5.2. Let G be a discrete amenable group acting on a von Neumann algebra N . Then an inclusion $N \subseteq N \rtimes G$ is crossed product-like by $\{\lambda_g\}_{g \in G}$.

Example 5.3. Let $A \subseteq B$ be a crossed product-like inclusion of C*-algebras acting non-degenerately on H . Then an inclusion $\bar{A}^w \subseteq \bar{B}^w$ of von Neumann algebras is crossed product-like.

Remark 5.4. Let $N \subseteq M$ be a crossed product-like inclusion of von Neumann algebras in $\mathbb{B}(H)$ by a discrete amenable group $U \subseteq \mathcal{N}_M(N)$. Then there is a left-invariant mean $m: \ell^\infty(U) \rightarrow \mathbb{R}$ with a net of finite subsets $\{F_\mu\} \subseteq U$ such that

$$\lim_\mu \frac{1}{|F_\mu|} \sum_{g \in F_\mu} f(g) = m(f), \quad f \in \ell^\infty(U).$$

Given a linear bounded map $\phi: U \rightarrow \mathbb{B}(H)$. For $\xi, \eta \in H$, define $\phi_{\xi, \eta} \in \ell^\infty(U)$ by

$$\phi_{\xi, \eta}(u) = \langle \phi(u)\xi, \eta \rangle, \quad u \in U.$$

Then there is an operator $T_\phi \in \mathbb{B}(H)$ which we will often write in the form

$$T_\phi = \int_{u \in U} \phi(u) dm$$

such that

$$\langle T_\phi \xi, \eta \rangle = m(\phi_{\xi, \eta}) = \int_{u \in U} \langle \phi(u) \xi, \eta \rangle dm, \quad \xi, \eta \in H.$$

By the construction of m , we have

$$\langle T_\phi \xi, \eta \rangle = \lim_{\mu} \frac{1}{|F_\mu|} \sum_{u \in F_\mu} \langle \phi(u) \xi, \eta \rangle, \quad \xi, \eta \in H,$$

that is, $T_\phi \in \overline{\text{conv}}^w \{\phi(u) : u \in U\}$. Furthermore,

$$\begin{aligned} \|T_\phi\| &= \sup_{\xi, \eta \in H} \left| \int_{u \in U} \langle \phi(u) \xi, \eta \rangle dm \right| \\ &\leq \int_{u \in U} \sup_{\xi, \eta \in H} |\langle \phi(u) \xi, \eta \rangle| dm = \int_{u \in U} \|\phi(u)\| dm. \end{aligned}$$

In the next lemma, we shall find a unital normal $*$ -homomorphism between von Neumann algebras. This lemma is originated in Christensen's work [6, Lemma 3.3], which discusses the perturbation theory for injective von Neumann algebras.

Lemma 5.5. *Let $N \subseteq M$ be an inclusion of von Neumann algebras in $\mathbb{B}(H)$ and let A, B be intermediate von Neumann subalgebras for $N \subseteq M$ with a normal conditional expectation $E: M \rightarrow B$. Suppose that $N \subseteq A$ is crossed product-like by a discrete amenable group U and $d(A, B) < \gamma < 1/4$. Then there exists a unital N -fixed normal $*$ -homomorphism $\Phi: A \rightarrow B$ such that*

$$\|\Phi - \text{id}_A\| \leq (8\sqrt{2} + 2)\gamma.$$

Proof. Let $A_0 := \text{span}\{xu : x \in N, u \in U\}$. By Stinespring's theorem, there exist a Hilbert space $\tilde{H} \supseteq H$ and a unital normal $*$ -homomorphism $\pi: M \rightarrow \mathbb{B}(\tilde{H})$ such that

$$E(x) = P_H \pi(x)|_H, \quad x \in M.$$

Let $m: \ell^\infty(U) \rightarrow \mathbb{R}$ be a left-invariant mean with a net of finite subsets $\{F_\mu\} \subseteq U$ such that

$$\lim_{\mu} \frac{1}{|F_\mu|} \sum_{g \in F_\mu} f(g) = m(f), \quad f \in \ell^\infty(U).$$

Define

$$t := \int_{u \in U} \pi(u) P_H \pi(u^*) dm.$$

Since $P_H \in \pi(N)'$, we have $t \in \pi(N)'$. Fix $x \in A_0$. Then there exist $\{u_1, \dots, u_N\} \subseteq U$ and $\{x_1, \dots, x_N\} \subseteq N_1$ such that $x = \sum_{i=1}^N x_i u_i$. For any $\xi, \eta \in H$,

$$\begin{aligned}
\langle \pi(x)t\xi, \eta \rangle &= \int_{u \in U} \langle \pi(x)\pi(u)P_H\pi(u^*)\xi, \eta \rangle dm \\
&= \int_{u \in U} \sum_{i=1}^N \langle \pi(x_i u_i u)P_H\pi(u^*)\xi, \eta \rangle dm \\
&= \sum_{i=1}^N \int_{v \in U} \langle \pi(x_i v)P_H\pi(v^* u_i)\xi, \eta \rangle d(u_i^* m) \\
&= \sum_{i=1}^N \int_{v \in U} \langle \pi(x_i v)P_H\pi(v^* u_i)\xi, \eta \rangle dm \\
&= \sum_{i=1}^N \int_{v \in U} \langle \pi(v)P_H\pi(v^* x_i u_i)\xi, \eta \rangle dm \\
&= \int_{u \in U} \langle \pi(u)P_H\pi(u^*)\pi(x)\xi, \eta \rangle dm = \langle t\pi(x)\xi, \eta \rangle.
\end{aligned}$$

Therefore, $t \in \pi(A)'$ by the normality of π . Furthermore, for $u \in U$, there is $v \in B_1$ such that $\|u - v\| < \gamma$. Then since $P_H \in \pi(B)'$, we have

$$\|\pi(u)P_H - P_H\pi(u)\| \leq \|\pi(u)P_H - \pi(v)P_H\| + \|P_H\pi(v) - P_H\pi(u)\| < 2\gamma.$$

Therefore,

$$\begin{aligned}
\|t - P_H\| &\leq \int_{u \in U} \|\pi(u)P_H\pi(u^*) - P_H\| dm \\
&= \int_{u \in U} \|\pi(u)P_H - P_H\pi(u)\| dm \leq 2\gamma < \frac{1}{2}.
\end{aligned}$$

Define $\delta := \|t - P_H\|$ and $q := \chi_{[1-\delta, 1]}(t)$. Since $\|q - P_H\| \leq 2\delta < 1$, there exists a unitary $w \in C^*(t, P_H, I_{\tilde{H}})$ such that $wP_H w^* = q$ and $\|w - I_{\tilde{H}}\| \leq 2\sqrt{2}\delta$ by Lemma 2.7 (3). Define a map $\Phi: A \rightarrow \mathbb{B}(\tilde{H})$ by

$$\Phi(x) := P_H w^* \pi(x) w|_H, \quad x \in A.$$

Since $t \in \overline{\text{conv}}^w \{\pi(u)P_H\pi(u^*) : u \in U\}$ and $P_H\pi(A)|_H = E(A) \subseteq B$, we have $\Phi(A) \subseteq B$. For any $x, y \in A$,

$$\begin{aligned}
\Phi(x)\Phi(y) &= P_H w^* \pi(x) w P_H w^* \pi(y) w P_H \\
&= P_H w^* \pi(x) q \pi(y) w P_H \\
&= P_H w^* q \pi(xy) w P_H \\
&= P_H w^* \pi(xy) w P_H = \Phi(xy).
\end{aligned}$$

Therefore, Φ is a $*$ -homomorphism.

Furthermore, for any $x \in A_1$,

$$\begin{aligned} \|\Phi(x) - x\| &\leq \|\Phi(x) - E(x)\| + \|E(x) - x\| \\ &\leq \|P_H w^* \pi(x) w P_H - P_H \pi(x) P_H\| + 2d(A, B) \\ &\leq 2\|w - I_{\tilde{H}}\| + 2d(A, B) \\ &\leq (8\sqrt{2} + 2)\gamma. \end{aligned}$$

Since $w \in C^*(t, P_H, I_{\tilde{H}}) \subseteq \pi(N)'$, Φ is a N -fixed map. \square

We base the next lemma on Christensen's work [6, Propositions 4.2 and 4.4], which show similar results for injective von Neumann algebras.

Lemma 5.6. *Let A, B and N be von Neumann algebras in $\mathbb{B}(H)$ with $N \subseteq A \cap B$. Suppose that $N \subseteq A$ is crossed product-like by a discrete amenable group U . Then given two unital N -fixed normal $*$ -homomorphisms $\Phi_1, \Phi_2: A \rightarrow B$ with $\|\Phi_1 - \Phi_2\| < 1$, there exists a unitary $u \in N' \cap B$ such that $\Phi_1 = \text{Ad}(u) \circ \Phi_2$ and $\|u - I\| \leq \sqrt{2}\|\Phi_1 - \Phi_2\|$.*

Proof. Let $A_0 := \text{span}\{xu : x \in N, u \in U\}$ and let $m: \ell^\infty(U) \rightarrow \mathbb{R}$ be a left-invariant mean with there is a net of finite subsets $\{F_\mu\} \subseteq U$ such that $m_\mu \rightarrow m$ in the weak- $*$ topology, where

$$m_\mu(f) = \frac{1}{|F_\mu|} \sum_{g \in F_\mu} f(g), \quad f \in \ell^\infty(U).$$

Define

$$s := \int_{u \in U} \Phi_1(u) \Phi_2(u^*) dm.$$

Since Φ_1 and Φ_2 are N -fixed maps and $U \subseteq \mathcal{N}_A(N)$, we have $s \in N' \cap B$. For $x \in A_0$, there exist $\{u_1, \dots, u_N\} \subseteq U$ and $\{x_1, \dots, x_N\} \subseteq N$ such that $x = \sum_{i=1}^N x_i u_i$. For any $\xi, \eta \in H$,

$$\begin{aligned} \langle \Phi_1(x) s \xi, \eta \rangle &= \int_{u \in U} \langle \Phi_1(x) \Phi_1(u) \Phi_2(u^*) \xi, \eta \rangle dm \\ &= \int_{u \in U} \sum_{i=1}^N \langle \Phi_1(c_i u_i u) \Phi_2(u^*) \xi, \eta \rangle dm \\ &= \sum_{i=1}^N \int_{v \in U} \langle \Phi_1(c_i v) \Phi_2(v^* u_i) \xi, \eta \rangle d(u_i^* m) \\ &= \sum_{i=1}^N \int_{v \in U} \langle \Phi_1(c_i v) \Phi_2(v^* u_i) \xi, \eta \rangle dm \\ &= \sum_{i=1}^N \int_{v \in U} \langle \Phi_1(v) \Phi_2(v^* c_i u_i) \xi, \eta \rangle dm \\ &= \int_{v \in U} \langle \Phi_1(v) \Phi_2(v^* x) \xi, \eta \rangle dm = \langle s \Phi_2(x) \xi, \eta \rangle. \end{aligned}$$

Therefore, by the normality of Φ_1 and Φ_2 ,

$$\Phi_1(x) s = s \Phi_2(x), \quad x \in A. \quad (31)$$

By taking adjoint,

$$s^* \Phi_1(x) = \Phi_2(x) s^*, \quad x \in A. \quad (32)$$

By (31) and (32), for $x \in A$, $s^* s \Phi_2(x) = s^* \Phi_1(x) s = \Phi_2(x) s^* s$. Thus,

$$|s|^{-1} \Phi_2(x) = \Phi_2(x) |s|^{-1}, \quad x \in A. \quad (33)$$

Furthermore,

$$\|s - I_H\| \leq \int_{u \in U} \|\Phi_1(u) \Phi_2(u^*) - \Phi_1(u) \Phi_1(u^*)\| dm \leq \|\Phi_1 - \Phi_2\| < 1.$$

Hence by Lemma 2.7 (1), we can choose the unitary $u \in C^*(s, I) \subseteq N' \cap B$ in the polar decomposition of s with $\|u - I\| \leq \sqrt{2}\|s - I\|$.

By (31) and (33),

$$\Phi_1(x)u = \Phi_1(x)s|s|^{-1} = s\Phi_2(x)|s|^{-1} = s|s|^{-1}\Phi_2(x) = u\Phi_2(x), \quad x \in A.$$

Therefore, $\Phi_1 = \text{Ad}(u) \circ \Phi_2$. □

Using Lemma 5.5 and 5.6, it follows Theorem C.

Theorem 5.7. *Let $N \subseteq M$ be an inclusion of von Neumann algebras in $\mathbb{B}(H)$ and let A, B be intermediate von Neumann subalgebras for $N \subseteq M$ with a normal conditional expectation from M onto B . Suppose that $N \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-2}$. Then there exists a unitary $u \in N' \cap (A \cup B)''$ such that $uAu^* = B$ and $\|u - I\| \leq 2(8 + \sqrt{2})\gamma$.*

Proof. By Lemma 5.5, there exists a unital N -fixed normal $*$ -homomorphism $\Phi: A \rightarrow B$ such that

$$\|\Phi - \text{id}_A\| \leq (8\sqrt{2} + 2)\gamma.$$

Since $(8\sqrt{2} + 2)\gamma < 1$, there exists a unitary $u \in N' \cap (A \cup B)''$ such that $\Phi = \text{Ad}(u)$ and $\|u - I\| \leq \sqrt{2}\|\Phi - \text{id}_A\|$ by Lemma 5.6. Thus,

$$uAu^* = \Phi(A) \subseteq B.$$

Fix $x \in B_1$. There exists $y \in A_1$ such that $\|x - y\| \leq \gamma$. Then,

$$\|y - uxu^*\| \leq \|y - x\| + \|x - uxu^*\| \leq \gamma + 2\|u - I\| < 1.$$

Thus, $d(uAu^*, B) < 1$, that is, $uAu^* = B$ by Proposition 2.4. □

Corollary 5.8. *Let $C \subseteq D$ be a unital inclusion of C*-algebras and let A, B be intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $d(A, B) < \gamma < 10^{-2}$. Then there exists a unitary $u \in (C^{**})' \cap W^*(A^{**}, B^{**})$ such that $uA^{**}u^* = B^{**}$ and $\|u - I\| \leq 2(8 + \sqrt{2})\gamma$.*

Proof. By a general construction, there exists a normal conditional expectation $E^{**}: D^{**} \rightarrow B^{**}$. Let (π, H) be the universal representation of D and identify A^{**}, B^{**}, C^{**} and D^{**} with $\pi(A)''$, $\pi(B)''$, $\pi(C)''$ and $\pi(D)''$, respectively. Then by Theorem 5.7 and Lemma 2.6, the corollary follows. □

6. UNITARY EQUIVALENCE

In this section, we show the fourth main result: Theorem D. For a unital inclusion $C \subseteq D$ of C^* -algebras acting on a separable Hilbert space H and sufficiently close separable intermediate C^* -subalgebras A, B for $C \subseteq D$ with a conditional expectation of D onto B , if $A = C \rtimes G$ with G discrete amenable and if $C' \cap A$ is weakly dense in $C' \cap \overline{A}^w$, then A and B are unitarily equivalent. The unitary can be chosen in the relative commutant of $C' \cap (A \cup B)''$. To show this, we modify the arguments of Section 5 in Christensen et al. [9].

Lemma 6.1. *Let $C \subseteq D$ be a unital inclusion of C^* -algebras acting non-degenerately on a separable Hilbert space H . Let A and B be separable intermediate C^* -subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $C' \cap C^*(A, B) \subseteq \overline{C' \cap A}^w$. If $d(A, B) < \gamma < 10^{-4}$, then for any finite subsets $X \subseteq B_1$, $Z_A \subseteq A_1$ and $\varepsilon, \mu > 0$, there exist finite subsets $Y \subseteq B_1$, $Z \subseteq A_1$, a positive constant $\delta > 0$, a unitary $u \in C' \cap C^*(A, B)$ and a C -fixed surjective $*$ -isomorphism $\theta: B \rightarrow A$ with the following conditions:*

- (i) $\delta < \varepsilon$;
- (ii) $X \subseteq_\varepsilon Y$;
- (iii) $\|u - I\| \leq 75\gamma$;
- (iv) $\theta \approx_{Y, \delta} \text{Ad}(u)$;
- (v) $\theta \approx_{X, 117\gamma} \text{id}_B$ and $\theta^{-1} \approx_{Z_A, 115\gamma} \text{id}_A$;
- (vi) For any C -fixed surjective $*$ -isomorphism $\phi: B \rightarrow A$ with $\phi^{-1} \approx_{Z, 365\gamma} \text{id}_A$, there exists a unitary $w \in C' \cap A$ such that

$$\text{Ad}(w) \circ \phi \approx_{Y, \delta/2} \theta \quad \text{and} \quad \|w - u\| \leq 665\gamma;$$

- (vii) For any finite subset $S \subseteq H_1$ and unitary $v \in C' \cap C^*(A, B)$ with $\text{Ad}(v) \approx_{Y, \delta} \theta$ and $\|v - u\| \leq 740\gamma$, there exists a unitary $\tilde{v} \in C' \cap A$ such that $\text{Ad}(\tilde{v}v) \approx_{X, \varepsilon} \theta$, $\|\tilde{v} - I\| \leq 740\gamma$ and

$$\|(\tilde{v}v - u)\xi\| < \mu \quad \text{and} \quad \|(\tilde{v}v - u)^*\xi\| < \mu, \quad \xi \in S.$$

Proof. Let X, Z_A, ε and μ be given. By Lemma 3.8, there exists a finite subset $Z_1 \subseteq B_1$ satisfying the following condition: given two unital C -fixed $*$ -homomorphisms $\phi_1, \phi_2: B \rightarrow B$ with $\phi_1 \approx_{Z_1, 32\gamma} \phi_2$, there exists a unitary $w_1 \in C' \cap B$ such that $\phi_1 \approx_{X, \varepsilon/3} \text{Ad}(w_1) \circ \phi_2$ and $\|w_1 - I_H\| \leq 32\sqrt{2}\gamma$.

By Proposition 4.2, there exists a C -fixed surjective $*$ -isomorphism $\beta: B \rightarrow A$ such that

$$\beta \approx_{Z_1, 17\gamma} \text{id}_B. \tag{34}$$

Let $X_0 := \beta(X)$.

By Lemma 3.12, there exist a finite set $Y_0 \subseteq A_1$ and $\delta > 0$ with the following properties: $\delta < \varepsilon/6$, $X_0 \subseteq Y_0$ and given a finite set $S_0 \subseteq H_1$ and a unitary $u \in C' \cap C^*(A, B)$ with $\|u - I\| \leq 740\gamma$ and

$$\|uy_0 - y_0u\| \leq 3\delta, \quad y_0 \in Y_0,$$

there exists a unitary $v \in C' \cap A$ such that $\|v - I\| \leq 740\gamma$,

$$\|vx_0 - x_0v\| \leq \frac{\varepsilon}{6}, \quad x_0 \in X_0$$

and

$$\|(v - u)\xi_0\| < \mu \quad \text{and} \quad \|(v - u)^*\xi_0\| < \mu, \quad \xi_0 \in S_0,$$

since $C' \cap C^*(A, B) \subseteq \overline{C' \cap A}^w$.

By Lemma 3.8, there exists a finite set $Z \subseteq A_1$ with the following properties: $\beta(Z_1) \cup Z_A \subseteq Z$ and given $\gamma_0 < 1/10$ and two unital C -fixed $*$ -homomorphism $\phi_1, \phi_2: A \rightarrow C^*(A, B)$ with $\phi_1 \approx_{Z, \gamma_0} \phi_2$, there exists a unitary $u_0 \in C' \cap C^*(A, B)$ such that $\text{Ad}(u_0) \circ \phi_1 \approx_{Y_0, \delta/2} \phi_2$ and $\|u_0 - I\| \leq \sqrt{2}\gamma_0$.

By Proposition 4.3, there exists a C -fixed surjective $*$ -isomorphism $\sigma: A \rightarrow B$ such that

$$\sigma \approx_{Z, 15\gamma} \text{id}_A \quad \text{and} \quad \sigma^{-1} \approx_{X, 17\gamma} \text{id}_B. \quad (35)$$

Hence, by the choice of Z , there exists a unitary $u_0 \in C' \cap C^*(A, B)$ such that

$$\sigma \approx_{Y_0, \delta/2} \text{Ad}(u_0) \quad (36)$$

and $\|u_0 - I\| \leq 15\sqrt{2}\gamma < 25\gamma$.

Since $\beta(Z_1) \subseteq Z$, (34) and (35), we have

$$\sigma \circ \beta \approx_{Z_1, 32\gamma} \text{id}_B. \quad (37)$$

By the definition of Z_1 , there exists a unitary $w_1 \in C' \cap B$ such that

$$\sigma \circ \beta \approx_{X, \varepsilon/3} \text{Ad}(w_1) \quad (38)$$

and $\|w_1 - I\| \leq 32\sqrt{2}\gamma < 50\gamma$.

Now define $\theta := \sigma^{-1} \circ \text{Ad}(w_1)$, $Y := \theta^{-1}(Y_0)$ and $u := u_0^* w_1$. Fix $y \in Y$. Let $y_0 := \theta(y) \in Y_0$. Then,

$$\begin{aligned} \|\theta(y) - \text{Ad}(u)(y)\| &= \|y_0 - (\text{Ad}(u) \circ \theta^{-1})(y_0)\| \\ &= \|y_0 - (\text{Ad}(u_0^*) \circ \sigma)(y_0)\| \\ &= \|\text{Ad}(u_0)(y_0) - \sigma(y_0)\| \leq \frac{\delta}{2}, \end{aligned}$$

since $\theta^{-1} = \text{Ad}(w_1^*) \circ \sigma$ and (36). Thus, $\theta \approx_{Y, \delta/2} \text{Ad}(u)$, so that condition (iv) holds.

By the definition of u , we have

$$\|u - I\| = \|w_1 - u_0\| \leq \|w_1 - I\| + \|I - u_0\| < 75\gamma.$$

Hence, condition (iii) holds.

For any $x \in X$,

$$\begin{aligned} \|\theta(x) - x\| &\leq \|(\sigma^{-1} \circ \text{Ad}(w_1))(x) - \sigma^{-1}(x)\| + \|\sigma^{-1}(x) - x\| \\ &\leq 2\|w_1 - I_H\| + 17\gamma \\ &\leq 100\gamma + 17\gamma = 117\gamma. \end{aligned} \quad (39)$$

For any $z \in Z$,

$$\begin{aligned} \|\theta^{-1}(z) - z\| &\leq \|(\text{Ad}(w_1^*) \circ \sigma)(z) - \text{Ad}(w_1^*)(z)\| + \|\text{Ad}(w_1^*)(z) - z\| \\ &\leq \|\sigma(z) - z\| + 2\|w_1 - I\| \\ &\leq 15\gamma + 100\gamma = 115\gamma. \end{aligned}$$

Therefore,

$$\theta^{-1} \approx_{Z, 115\gamma} \text{id}_A. \quad (40)$$

Since $Z_A \subseteq Z$, we have $\theta^{-1} \approx_{Z_A, 115\gamma} \text{id}_A$, so that condition (v) holds.

By (38),

$$\theta = \sigma^{-1} \circ \text{Ad}(w_1) \approx_{X, \varepsilon/3} \sigma^{-1} \circ \sigma \circ \beta = \beta \quad (41)$$

Fix $x_0 \in X_0$. Let $x := \beta^{-1}(x_0) \in X$. Then, by (41),

$$\|\theta^{-1}(x_0) - \beta^{-1}(x_0)\| = \|(\theta^{-1} \circ \beta)(x) - x\| = \|\beta(x) - \theta(x)\| \leq \frac{\varepsilon}{3}.$$

Therefore,

$$\theta^{-1} \approx_{X_0, \varepsilon/3} \beta^{-1}. \quad (42)$$

Hence,

$$X = \beta^{-1}(X_0) \subseteq_{\varepsilon/3} \theta^{-1}(X_0) \subseteq \theta^{-1}(Y_0) = Y,$$

so that condition (ii) holds.

We now verify condition (vi). Let $\phi: B \rightarrow A$ be a C -fixed surjective $*$ -isomorphism with $\phi^{-1} \approx_{Z, 365\gamma} \text{id}_A$. By (35),

$$\phi^{-1} \approx_{Z, 380\gamma} \sigma.$$

Thus, by the definition of Z , there exists a unitary $w_0 \in C' \cap B$ such that

$$\text{Ad}(w_0) \circ \phi^{-1} \approx_{Y_0, \delta/2} \sigma \quad (43)$$

and $\|w_0 - I\| \leq 380\sqrt{2}\gamma < 540\gamma$. Fix $y \in Y$. Let $y_0 := \theta(y) \in Y_0$. Then, since $w_0^*w_1 \in B$, we have

$$\begin{aligned} \|\theta(y) - (\text{Ad}(\phi(w_0^*w_1)) \circ \phi)(y)\| &= \|\theta(y) - (\phi \circ \text{Ad}(w_0^*w_1))(y)\| \\ &= \|y_0 - (\phi \circ \text{Ad}(w_0^*) \circ \sigma)(y_0)\| \\ &= \|(\text{Ad}(w_0) \circ \phi^{-1})(y_0) - \sigma(y_0)\| \leq \frac{\delta}{2} \end{aligned}$$

by (43). Define $w := \phi(w_0^*w_1)$ so $\theta \approx_{Y, \delta/2} \text{Ad}(w) \circ \phi$. Since ϕ is C -fixed map and $w_0, w_1 \in C'$, w is in $C' \cap A$. Moreover,

$$\begin{aligned} \|w - u\| &\leq \|w - I\| + \|I - u\| \leq \|w_0^*w_1 - I\| + 75\gamma \\ &\leq \|w_0 - I\| + \|I - w_1\| + 75\gamma \leq (540 + 50 + 75)\gamma \\ &= 665\gamma. \end{aligned}$$

Therefore, condition (vi) is proved.

It only remains to prove condition (vii). Let $S \subseteq H_1$ be a finite set and $v \in C' \cap C^*(A, B)$ be a unitary with $\|v - u\| \leq 740\gamma$ and

$$\text{Ad}(v) \approx_{Y, \delta} \theta. \quad (44)$$

Fix $y_0 \in Y_0$. Let $y := \theta^{-1}(y_0) \in Y$. Then,

$$\begin{aligned} \|\sigma(y_0) - \text{Ad}(w_1v^*)(y_0)\| &= \|(\text{Ad}(w_1^*) \circ \sigma)(y_0) - \text{Ad}(v^*)(y_0)\| \\ &= \|y - (\text{Ad}(v^*) \circ \theta)(y)\| \\ &= \|\text{Ad}(v)(y) - \theta(y)\| \leq \delta. \end{aligned} \quad (45)$$

This and (36) give $\text{Ad}(u_0) \approx_{Y_0, 3\delta/2} \text{Ad}(w_1v^*)$. Therefore, for any $y_0 \in Y_0$,

$$\|(v w_1^* u_0) y_0 - y_0 (v w_1^* u_0)\| = \|u_0 y_0 u_0^* - (w_1 v^*) y_0 (w_1 v^*)^*\| \leq \frac{3}{2} \delta. \quad (46)$$

Furthermore,

$$\|vw_1^*u_0 - I\| = \|w_1^*u_0 - v^*\| = \|u^* - v^*\| \leq 740\gamma. \quad (47)$$

Let $S_0 := S \cup w_1S \cup vS$. By the definition of Y_0 and δ , with $vw_1^*u_0$ and S_0 , there exists a unitary $v_0 \in C' \cap A$ such that $\|v_0 - I\| \leq 740\gamma$,

$$\|v_0x_0 - x_0v_0\| \leq \frac{\varepsilon}{6}, \quad x_0 \in X_0 \quad (48)$$

and

$$\|(v_0 - vw_1^*u_0)\xi_0\| < \mu \quad \text{and} \quad \|(v_0 - vw_1^*u_0)^*\xi_0\| < \mu, \quad \xi_0 \in S_0. \quad (49)$$

Let $\tilde{v} := v_0^*$. Then, $\|\tilde{v} - I\| \leq \|v_0 - I\| \leq 740\gamma$.

For any $\xi \in S$,

$$\begin{aligned} \|(\tilde{v}v - u)\xi\| &= \|(v_0^*v - u_0^*w_1)\xi\| = \|(v_0^* - u_0^*w_1v^*)v\xi\| \\ &= \|(v_0 - vw_1u_0)^*v\xi\| < \mu \end{aligned}$$

by (49) and $v\xi \in S_0$. Moreover,

$$\|(\tilde{v}v - u)^*\xi\| = \|(v_0^*v - u_0^*w_1)^*\xi\| = \|v^*(v_0 - vw_1^*u_0)\xi\| < \mu$$

by (49).

For any $x_0 \in X_0$, by (45) and (48),

$$\begin{aligned} \|\theta^{-1}(x_0) - \text{Ad}(v^*v_0)(x_0)\| &\leq \|\theta^{-1}(x_0) - \text{Ad}(v^*)(x_0)\| + \|\text{Ad}(v^*)(x_0) - \text{Ad}(v^*v_0)(x_0)\| \\ &= \|(\text{Ad}(w_1^*) \circ \sigma)(x_0) - \text{Ad}(v^*)(x_0)\| + \|x_0 - \text{Ad}(v_0)(x_0)\| \\ &= \|\sigma(x_0) - \text{Ad}(w_1v^*)(x_0)\| + \|v_0x_0 - x_0v_0\| \\ &< \delta + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{3}. \end{aligned} \quad (50)$$

Let $x \in X$ and $x_0 := \beta(x) \in X_0$. By (41), (42) and (50),

$$\begin{aligned} \|\text{Ad}(\tilde{v}v)(x) - \theta(x)\| &\leq \|\text{Ad}(\tilde{v}v)(x) - \beta(x)\| + \|\beta(x) - \theta(x)\| \\ &\leq \|(\text{Ad}(\tilde{v}v) \circ \beta^{-1})(x_0) - x_0\| + \frac{\varepsilon}{3} \\ &= \|\beta^{-1}(x_0) - \text{Ad}(v^*v_0)(x_0)\| + \frac{\varepsilon}{3} \\ &\leq \|\beta^{-1}(x_0) - \theta^{-1}(x_0)\| + \|\theta^{-1}(x_0) - \text{Ad}(v^*v_0)(x_0)\| + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, condition (vii) holds. \square

Lemma 6.2. *Let $C \subseteq D$ be a unital inclusion of C*-algebras acting non-degenerately on a separable Hilbert space H . Let A and B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{\xi_n\}_{n=0}^\infty$ be dense subsets in A_1 , B_1 and H_1 , respectively. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $C' \cap C^*(A, B) \subseteq \overline{C' \cap A}^w$. If $d(A, B) < \gamma < 10^{-5}$, then there exist finite subsets $\{X_n\}_{n=0}^\infty, \{Y_n\}_{n=0}^\infty \subseteq B_1$, $\{Z_n\}_{n=0}^\infty \subseteq A_1$, positive constants $\{\delta_n\}_{n=0}^\infty$, unitaries $\{u_n\}_{n=0}^\infty \subseteq C' \cap C^*(A, B)$ and C -fixed surjective *-isomorphisms $\{\theta_n: B \rightarrow A\}_{n=0}^\infty$ with the following conditions:*

- (1) For $n \geq 1$, $b_1, \dots, b_n \in X_n$;
- (2) For $n \geq 0$, $X_n \subseteq_{2^{-n}/3} Y_n$ and $\delta_n < 2^{-n}$;
- (3) For $n \geq 1$, $\theta_n \approx_{X_{n-1}, 2^{-(n-1)}} \theta_{n-1}$;
- (4) For $n \geq 0$, $\theta_n \approx_{Y_n, \delta_n} \text{Ad}(u_n)$;
- (5) For $1 \leq j \leq n$, $\|(u_n - u_{n-1})\xi_j\| < 2^{-n}$ and $\|(u_n - u_{n-1})^*\xi_j\| < 2^{-n}$;
- (6) For $1 \leq j \leq n$, there exists $x \in X_n$ such that $\|\theta_n(x) - b_j\| \leq 9/10$;
- (7) For $n \geq 0$ and a C -fixed surjective $*$ -isomorphism $\phi: B \rightarrow A$ with $\phi^{-1} \approx_{Z_n, 365\gamma} \text{id}_A$, there exists a unitary $w \in C' \cap A$ such that $\text{Ad}(w) \circ \phi \approx_{Y_n, \delta_n/2} \theta_n$ and $\|w - u_n\| \leq 665\gamma$;
- (8) For $n \geq 0$, a finite subset $S \subseteq H_1$ and a unitary $v \in C' \cap C^*(A, B)$ with $\text{Ad}(v) \approx_{Y_n, \delta_n} \theta_n$ and $\|v - u_n\| \leq 740\gamma$, there exists a unitary $\tilde{v} \in C' \cap A$ such that $\text{Ad}(\tilde{v}v) \approx_{X_n, 2^{-(n+1)}} \theta_n$, $\|\tilde{v} - I\| \leq 740\gamma$ and

$$\|(\tilde{v}v - u_n)\xi\| < \frac{1}{2^{n+1}} \quad \text{and} \quad \|(\tilde{v}v - u_n)^*\xi\| < \frac{1}{2^{n+1}}, \quad \xi \in S;$$

- (9) For $n \geq 0$, there is a unitary $z \in A$ such that $\|z - u_n\| \leq 75\gamma$.

Proof. We prove this lemma by using the induction. Denote by $(a)_n$ the condition (a) for n . Let $X_0 = Y_0 = Z_0 = \emptyset$, $\delta_0 = 1/2$, $u_0 = I$ and $\theta_0: B \rightarrow A$ be any C -fixed surjective $*$ -isomorphism by Proposition 4.2. Then, conditions $(1)_0$, $(3)_0$, $(5)_0$ and $(6)_0$ do not be defined. Conditions $(2)_0$ and $(4)_0$ are clear, since $X_0 = Y_0 = \emptyset$. In conditions $(7)_0$, $(8)_0$ and $(9)_0$, by taking $w = I$, $\tilde{v} = v^*$ and $z = I$, that conditions are satisfied.

Assume the statement holds for n ; we will prove it for $n+1$. By $(9)_n$, there exists a unitary $z \in A$ such that $\|z - u_n\| \leq 75\gamma$. For $1 \leq j \leq n+1$, there exists $x_j \in B_1$ such that $\|x_j - z^*a_jz\| \leq \gamma$. Define $X_{n+1} := X_n \cup Y_n \cup \{b_n\} \cup \{x_1, \dots, x_{n+1}\}$. In Lemma 6.1, let $X = X_{n+1}$, $Z_A = Z_n$, $\varepsilon = \delta_n/6$ and $\mu = 2^{-(n+2)}$ and so there exist $Y_{n+1} \subseteq B_1$, $Z_{n+1} \subseteq A_1$, $\delta_{n+1} > 0$, $u \in C' \cap C^*(A, B)$ and $\theta: B \rightarrow A$ with conditions (i)-(vii) of that lemma. By Lemma 6.1 (i), $\delta_{n+1} < \varepsilon = \delta_n/6 < 2^{-(n+1)}/3$. By Lemma 6.1 (ii), $X_{n+1} \subseteq_{2^{-(n+1)}/3} Y_{n+1}$. Thus, condition $(2)_{n+1}$ holds.

By applying θ to condition $(7)_n$, we may find a unitary $w \in C' \cap A$ such that

$$\text{Ad}(w) \circ \theta \approx_{Y_n, \delta_n/2} \theta_n \tag{51}$$

and $\|w - u_n\| \leq 665\gamma$.

Fix $y \in Y_n$. Since $Y_n \subseteq X_{n+1} \subseteq_{\delta_n/6} Y_{n+1}$, there exists $\tilde{y} \in Y_{n+1}$ such that $\|y - \tilde{y}\| \leq \delta_n/6$. Then, by Lemma 6.1 (iv),

$$\begin{aligned} \|\text{Ad}(u)(y) - \theta(y)\| &\leq \|\text{Ad}(u)(y) - \text{Ad}(u)(\tilde{y})\| + \|\text{Ad}(u)(\tilde{y}) - \theta(\tilde{y})\| + \|\theta(\tilde{y}) - \theta(y)\| \\ &\leq \frac{\delta_n}{6} + \delta_{n+1} + \frac{\delta_n}{6} \leq \frac{\delta_n}{2}. \end{aligned}$$

This and (51) give

$$\text{Ad}(wu) \approx_{Y_n, \delta_n} \theta_n. \tag{52}$$

Moreover,

$$\|wu - u_n\| \leq \|w(u - I)\| + \|w - u_n\| \leq 75\gamma + 665\gamma = 740\gamma. \tag{53}$$

By (52) and (53), we can apply wu and $\{\xi_1, \dots, \xi_{n+1}\}$ to condition $(8)_n$. Hence, there exists a unitary $\tilde{v} \in C' \cap A$ such that

$$\text{Ad}(\tilde{v}wu) \approx_{X_n, 2^{-(n+1)}} \theta_n, \tag{54}$$

$\|\tilde{v} - I\| \leq 740\gamma$ and

$$\|(\tilde{v}wu - u_n)\xi_j\| < \frac{1}{2^{n+1}} \quad \text{and} \quad \|(\tilde{v}wu - u_n)^*\xi_j\| < \frac{1}{2^{n+1}}, \quad 1 \leq j \leq n+1. \quad (55)$$

Define $\theta_{n+1} := \text{Ad}(\tilde{v}w) \circ \theta$ and $u_{n+1} := \tilde{v}wu$. By (55), condition (5)_{n+1} is trivial. Since $\tilde{v}w \in A$ and

$$\|\tilde{v}w - u_{n+1}\| = \|\tilde{v}w - \tilde{v}wu\| = \|I - u\| \leq 75\gamma,$$

condition (9)_{n+1} holds.

By Lemma 6.1 (iv), $\theta_{n+1} = \text{Ad}(\tilde{v}w) \circ \theta \approx_{Y_{n+1}, \delta_{n+1}} \text{Ad}(\tilde{v}wu) = \text{Ad}(u_{n+1})$. Thus, condition (4)_{n+1} is satisfied.

Fix $x \in X_n$. Let $y \in Y_{n+1}$ satisfy $\|x - y\| \leq 2^{-(n+1)}/3$. Then, by (4)_{n+1},

$$\begin{aligned} & \|\theta_{n+1}(x) - \text{Ad}(u_{n+1})(x)\| \\ & \leq \|\theta_{n+1}(x) - \theta_{n+1}(y)\| + \|\theta_{n+1}(y) - \text{Ad}(u_{n+1})(y)\| + \|\text{Ad}(u_{n+1})(y) - \text{Ad}(u_{n+1})(x)\| \\ & \leq \frac{1}{3 \cdot 2^{n+1}} + \delta_{n+1} + \frac{1}{3 \cdot 2^{n+1}} < \frac{1}{2^{n+1}}. \end{aligned}$$

Therefore,

$$\theta_{n+1} \approx_{X_n, 2^{-(n+1)}} \text{Ad}(u_{n+1}).$$

This and (54) give $\theta_{n+1} \approx_{X_n, 2^{-n}} \theta_n$. Hence, condition (3)_{n+1} holds.

For any $x \in A_1$,

$$\begin{aligned} & \|\text{Ad}(\tilde{v}w)(x) - \text{Ad}(z)(x)\| \\ & \leq \|\text{Ad}(\tilde{v}w)(x) - \text{Ad}(w)(x)\| + \|\text{Ad}(w)(x) - \text{Ad}(u_n)(x)\| + \|\text{Ad}(u_n)(x) - \text{Ad}(z)(x)\| \\ & \leq 2\|\tilde{v} - I\| + 2\|w - u_n\| + 2\|u_n - z\| \\ & \leq (1480 + 1330 + 150)\gamma = 2960\gamma. \end{aligned} \quad (56)$$

For $1 \leq j \leq n+1$, there exists $x_j \in X_{n+1}$ such that $\|x_j - z^*a_jz\| \leq \gamma$ by the definition of X_{n+1} . (56) and Lemma 6.1 (v) give

$$\begin{aligned} & \|\theta_{n+1}(x_j) - a_j\| \\ & \leq \|\theta_{n+1}(x_j) - \text{Ad}(\tilde{v}w)(x_j)\| + \|\text{Ad}(\tilde{v}w)(x_j) - \text{Ad}(z)(x_j)\| + \|\text{Ad}(z)(x_j) - a_j\| \\ & \leq \|(\text{Ad}(\tilde{v}w) \circ \theta)(x_j) - \text{Ad}(\tilde{v}w)(x_j)\| + 2960\gamma + \|x_j - z^*b_jz\| \\ & \leq \|\theta(x_j) - x_j\| + 2960\gamma + \gamma \\ & \leq 3078\gamma < \frac{9}{10}. \end{aligned}$$

Therefore, condition (6)_{n+1} is proved.

Let $\phi: B \rightarrow A$ be a C -fixed surjective $*$ -isomorphism with $\phi^{-1} \approx_{Z_{n+1}, 365\gamma} \text{id}_A$. By Lemma 6.1 (vi), there exists a unitary $\tilde{w} \in C' \cap A$ such that

$$\text{Ad}(\tilde{w}) \circ \phi \approx_{Y_{n+1}, \delta_{n+1}/2} \theta \quad (57)$$

and $\|\tilde{w} - u\| \leq 665\gamma$. For any $y \in Y_{n+1}$, by (57),

$$\|(\text{Ad}(\tilde{v}w\tilde{w}) \circ \phi)(y) - \theta_{n+1}(y)\| = \|(\text{Ad}(\tilde{w}) \circ \phi)(y) - \theta(y)\| \leq \frac{\delta_{n+1}}{2}.$$

Furthermore, we have

$$\|\tilde{v}w\tilde{w} - u_{n+1}\| = \|\tilde{v}w\tilde{w} - \tilde{v}wu\| = \|\tilde{w} - u\| \leq 665\gamma.$$

Thus, $\tilde{v}v\tilde{w}$ satisfies (7) $_{n+1}$.

It remains to prove condition (8) $_{n+1}$. Let $S \subseteq H_1$ be a finite set and $v \in C' \cap C^*(A, B)$ be a unitary with $\|v - u_{n+1}\| \leq 740\gamma$ and $\text{Ad}(v) \approx_{Y_{n+1}, \delta_{n+1}} \theta_{n+1}$. Then, we have

$$\|w^* \tilde{v}^* v - u\| = \|v - \tilde{v}wu\| = \|v - u_{n+1}\| \leq 740\gamma$$

and $\text{Ad}(w^* \tilde{v}^* v) \approx_{Y_{n+1}, \delta_{n+1}} \text{Ad}(w^* \tilde{v}^*) \circ \theta_{n+1} = \theta$. Hence, by applying Lemma 6.1 (vii) to $w^* \tilde{v}^* v$ and $S' := S \cup \{w^* \tilde{v}^* \xi : \xi \in S\}$, there exists a unitary $v' \in C' \cap A$ such that $\text{Ad}(v' w^* \tilde{v}^* v) \approx_{X_{n+1}, \delta_n/6} \theta$, $\|v' - I\| \leq 740\gamma$ and

$$\|(v' w^* \tilde{v}^* v - u)\xi'\| < \frac{1}{2^{n+2}} \quad \text{and} \quad \|(v' w^* \tilde{v}^* v - u)^* \xi'\| < \frac{1}{2^{n+2}}, \quad \xi' \in S'.$$

For any $x \in X_n$, we have

$$\|\text{Ad}(\tilde{v}wv'w^*\tilde{v}^*v)(x) - \theta_{n+1}(x)\| = \|\text{Ad}(v'w^*\tilde{v}^*v)(x) - \theta(x)\| \leq \frac{\delta_n}{6} < \frac{1}{2^{n+2}}.$$

and

$$\|\tilde{v}wv'w^*\tilde{v}^* - I\| = \|v' - I\| \leq 740\gamma.$$

For $\xi \in S$, we have

$$\|(\tilde{v}wv'w^*\tilde{v}^*v - u_{n+1})\xi\| = \|(v'w^*\tilde{v}^*v - u)\xi\| < \frac{1}{2^{n+2}}.$$

and

$$\|(\tilde{v}wv'w^*\tilde{v}^*v - u_{n+1})^* \xi\| = \|(v'w^*\tilde{v}^*v - u)^* w^* \tilde{v}^* \xi\| < \frac{1}{2^{n+2}}.$$

Therefore, $\tilde{v}wv'w^*\tilde{v}^*$ satisfies (8) $_{n+1}$, and the lemma follows. \square

Proposition 6.3. *Let $C \subseteq D$ be a unital inclusion of C^* -algebras acting non-degenerately on a separable Hilbert space H . Let A and B be separable intermediate C^* -subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $C' \cap C^*(A, B) \subseteq \overline{C'} \cap \overline{A}^w$. If $d(A, B) < 10^{-5}$, then there exists a unitary $u \in C' \cap (A \cup B)''$ such that $uAu^* = B$.*

Proof. Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{\xi_n\}_{n=0}^\infty$ be dense subsets in A_1 , B_1 and H_1 , respectively. In Lemma 6.2, we may choose $\{X_n\}_{n=0}^\infty$, $\{Y_n\}_{n=0}^\infty$, $\{Z_n\}_{n=0}^\infty$, $\{\delta_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$ and $\{\theta_n\}_{n=0}^\infty$ with (1)–(8). For any b_k and $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $2^{-(N-1)} < \varepsilon$ and $k < N$. For $m \geq n \geq N$,

$$\|\theta_m(b_k) - \theta_n(b_k)\| \leq \sum_{j=n}^{m-1} \|\theta_{j+1}(b_k) - \theta_j(b_k)\| \leq \sum_{j=n}^{m-1} \frac{1}{2^j} < \frac{1}{2^{N-1}} < \varepsilon.$$

Thus, for any b_k , $\{\theta_n(b_k)\}_{n=0}^\infty$ is a Cauchy sequence. Since $\|\theta_n\| \leq 1$, the sequence $\{\theta_n\}$ converges to a C -fixed $*$ -homomorphism $\theta: B \rightarrow A$ in the point-norm topology.

For any a_j and $n \geq j$, there is $x \in X_n$ such that $\|\theta_n(x) - a_j\| \leq 9/10$.

$$\begin{aligned} \|a_j - \theta(x)\| &\leq \|a_j - \theta_n(x)\| + \sum_{m=n}^{\infty} \|\theta_{m+1}(x) - \theta_m(x)\| \\ &\leq \frac{9}{10} + \sum_{m=n}^{\infty} \frac{1}{2^m} \leq \frac{9}{10} + \frac{1}{2^{n-1}}. \end{aligned}$$

Since $n \geq j$ was arbitrary and $\{a_n\}$ is dense set in A_1 , we have $d(A, \theta(B)) < 1$. Therefore, θ is surjective by Corollary 2.4.

By Lemma 6.2 (5), $\{u_n\}$ converges to a unitary $u \in C' \cap (A \cup B)''$ in the $*$ -strong topology. Moreover, by Lemma 6.2 (4), we have $\theta = \text{Ad}(u)$. Therefore, $A = uBu^*$, since θ is surjective. \square

Finally, we show Theorem D by using Proposition 6.3 and Corollary 5.8.

Theorem 6.4. *Let $C \subseteq D$ be a unital inclusion of C*-algebras acting on a separable Hilbert space H . Let A and B be separable intermediate C*-subalgebras for $C \subseteq D$ with a conditional expectation $E: D \rightarrow B$. Suppose that $C \subseteq A$ is crossed product-like by a discrete amenable group and $C' \cap A$ is weakly dense in $C' \cap \overline{A}^w$. If $d(A, B) < 10^{-7}$, then there exists a unitary $u \in C' \cap (A \cup B)''$ such that $uAu^* = B$.*

Proof. Let $d(A, B) < \gamma < 10^{-7}$. By Corollary 5.8, there exists a unitary $u_0 \in (C^{**})' \cap W^*(A^{**}, B^{**})$ such that $u_0 A^{**} u_0^* = B^{**}$ and $\|u_0 - I\| \leq 19\gamma$.

Let e_D be the support projection of D and define $K := \text{ran}(e_D) \subseteq H$. Now restrict A, B, C and D to K . By the universal property, there exists a unique normal representation $\pi: D^{**} \rightarrow \mathbb{B}(K)$ such that $\pi|_D = \text{id}_D$ and $\pi(D^{**}) = D''$.

Define $\tilde{A} := \pi(u_0)A\pi(u_0^*) \subseteq \mathbb{B}(K)$, then $d(\tilde{A}, B) \leq 2\|u_0 - I\| + d(A, B) < 39\gamma < 10^{-5}$. Since $\tilde{A}'' = \pi(u_0)\pi(A^{**})\pi(u_0^*) = \pi(B^{**}) = B''$ and $C' \cap A$ is weakly dense in $C' \cap \overline{A}^w$,

$$C' \cap C^*(\tilde{A}, B) \subseteq C' \cap \tilde{A}'' = \pi(u_0)(C' \cap A'')\pi(u_0^*) = \pi(u_0)(\overline{C' \cap A}^w)\pi(u_0^*) = \overline{C' \cap A}^w.$$

Therefore, there exists a unitary $u_1 \in C' \cap B'' \subseteq \mathbb{B}(K)$ such that $u_1 \tilde{A} u_1^* = B$ by Proposition 6.3. Hence, the unitary u is given by

$$u = u_1 \pi(u_0) + (I_K - e_D) \in C' \cap (A \cup B)'' \subseteq \mathbb{B}(H),$$

so that $uAu^* = B$. \square

Example 6.5. Let $C = C(\mathbb{T})$ and $A = C(\mathbb{T}) \rtimes \mathbb{Z}$ act on $H = \mathcal{L}^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$. Then we have $C' \cap A = C$ and $C' \cap \overline{A}^w = \mathcal{L}^\infty(\mathbb{T})$, that is, $C' \cap A$ is weakly dense in $C' \cap \overline{A}^w$.

But we should be careful that $C' \cap \overline{A}^w$ may not be equal to the weak closure of $C' \cap A$ in general.

Example 6.6. Let α be a free action of a group G on a simple C*-algebra C and $A = C \rtimes_\alpha G$ act irreducibly on a Hilbert space H . Then $C' \cap A = \mathbb{C}$ but $C' \cap \overline{A}^w = C' \cap \mathbb{B}(H)$.

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